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Plasma Physics via Vlasov simulations

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A path to the Vlasov equation

How to study such a complex system ? **A plasma is a globally neutral collection of discrete charged particles behaving as a collective system dominated by long-range electromagnetic forces**

Often….a plasma is defined as a "high temperature gas"

But remember: a plasma is NOT a gas

Why ? because

1. collisions are in general inefficient

2. A plasma is a long range interactions system

WE NEED A MATHEMATICAL MODEL

Too many particles for a N-body description even for modern super-computing systems

TOWARDS A CONTINUUM MODEL…

The statistical description of a **^N particles plasma** *is based on the* **probability densities ^F** *giving the probability of finding simultaneously the particles at locations (x¹ ,.., x^N , v¹ ,..,v^N) in phase space. Too much complicate!*

The probability $f_1(x_1,...,x_s,v_1,...v_s)$ of finding particles $1,.., s$ at location $(\mathrm{x}_1...,\mathrm{x}_\mathrm{s},\!\mathrm{v}_1,...\mathrm{v}_\mathrm{s})$ is given by integrating the d.f. allover the particles except 1 to s:

$$
f_1(x_1, \ldots, x_s, v_1, \ldots, v_s) = \int F(x_1, \ldots, x_N, v_1, \ldots, v_N) dx_{s+1}, \ldots, dx_N dv_{s+1}, \ldots, dv_N dv_{s+1}
$$

The probability density F contains the effects of the interactions among particles

When the interaction potential can be neglected, the particles can be considered as **statistically independent.** *For example:*

$$
F_2(x_1, x_2) = F_1(x_1) F_1(x_2)
$$

When instead the interaction potential among particles is present, the probability densities can be written trough a cluster expansion:

$$
F_2(x_1, x_2) = F_1(x_1) F_1(x_2) [1 + P_{12}(x_1, x_2)]
$$

and so on for **Fⁱ , i > 2**

P12 : *two particle correlation function*

In general, single particle interactions are assumed as negligible

 $\mathbf{P}_{12} \ll 1$ (and so on)

STARTING POINT: THE MANY BODY D.F. OBEYS THE LIOUVILLE EQUATION

$$
\frac{\partial F}{\partial t} + \sum_{i} \left(\frac{\partial F}{\partial \mathbf{x}_{i}} \cdot \mathbf{v}_{i} + \frac{\partial F}{\partial \mathbf{v}_{i}} \cdot \mathbf{a}_{i}^{T} \right) = 0
$$

where $\boldsymbol{a}_{i}^{\text{T}}$ is the TOTAL acceleration of the i-particle due to inter-particle *interactions and external forces.*

$$
F =
$$
 hyper-particle with coordinate $(\mathbf{x}_1, ..., \mathbf{x}_N, \mathbf{v}_1, ..., \mathbf{v}_N)$

The value of F, the probability density, **REMAINS CONSTANT AT THE LOCATION OF THE HYPER-PARTICLE***. But this does not mean that F is constant in time at any fixed point* $(\mathbf{x}_1, ..., \mathbf{x}_N, \mathbf{v}_1, ..., \mathbf{v}_N)$ *By integrating the Liouville equation over all the space – velocity coordinates* but one particle, we get the differential equation for **the one particle d.f.**

> *we need to simplify! we discuss later when this approach is reasonable…*

$$
\frac{\partial f^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \mathbf{x}_1} + \int \mathbf{a}_1^{\mathrm{T}} \cdot \frac{\partial F}{\partial \mathbf{v}_1} \, \mathrm{d}\mathbf{x}_2 \dots \mathrm{d}\mathbf{x}_N \mathrm{d}\mathbf{v}_2 \dots \mathrm{d}\mathbf{v}_N
$$

where we have assumed that $\mathbf{a}_i^T \cdot \mathbf{e}_l$ does not depend \mathbf{v}_l (i >1)

Here we have used the fact that the net flux of particles out of the system vanishes

$$
\int d\mathbf{x}_2 \frac{\partial}{\partial \mathbf{x}_2} \cdot \int \mathbf{v}_2 F d\mathbf{v}_2 = 0
$$

(constant total
particle number)

 $\int d\mathbf{v}_2$ ∂F ∂ **v**₂ $= 0$

with the condition $F(v \rightarrow \pm \infty) = 0$

as well as

In general, the force acting on the ith-particle includes both external and inter-particle forces:

$$
\mathbf{a}_i^T(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{v}_1, \dots, \mathbf{v}_N) = (\mathbf{q}_i/\mathbf{m}_i) \left[\mathbf{E} + (\mathbf{v}_i \times \mathbf{B})/c\right] + \sum_j \mathbf{a}_{ij}
$$
\n
$$
\mathbf{a}_{\text{CC}} \cdot \mathbf{e}_{\text{KL}} + \sum_{\text{acc}} \mathbf{a}_{\text{CL}} \cdot \mathbf{e}_{\text{KL}} + \sum_{\text{acc}} \mathbf{e}_{\text{KL}} \cdot \mathbf{e}_{\text{KL}} + \sum_{\text{adv}} \mathbf{e}_{\text{KL}} \cdot \mathbf{e}_{\text{KL}} + \sum_{\text{adv}} \mathbf{e}_{\text{KL}} \cdot \mathbf{e}_{\text{KL}})
$$

THE ACCELERATION TERM BECOMES:

$$
\mathbf{a}_{1}^{ext} \cdot \frac{\partial f^{(1)}}{\partial \mathbf{v}_{1}} + \int \sum_{j} \mathbf{a}_{1j} \cdot \frac{\partial F}{\partial \mathbf{v}_{1}} d\mathbf{x}_{2} \dots d\mathbf{x}_{N} d\mathbf{v}_{2} \dots d\mathbf{v}_{N} = 0
$$

because
$$
\int \mathbf{a}_{k>1}^T \cdot \frac{\partial F}{\partial \mathbf{v}_k} d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N = 0
$$

since $\mathbf{a}_i \cdot \mathbf{e}_1$ does not depend \mathbf{v}_1 so that, as above, $\int d\mathbf{v}_{k>1}$ ∂F $\widehat{c} \mathbf{v}^{}_{k}$ $= 0$ *(non relativistic limit for the* wB *term)*

the inter particle force term

 $\mathbf{a}_{1 \texttt{j}}^{}$. ∂F $\partial \mathbf{v}_1^{}$ $\mathbf{d}\mathbf{x}_2...\mathbf{d}\mathbf{x}_N\mathbf{d}\mathbf{v}_2...\mathbf{d}\mathbf{v}_N$ *j* $\int\! \sum$

reduces to

$$
\sum_{\alpha} \int \mathbf{a}_{1\alpha} \cdot \frac{\partial}{\partial \mathbf{v}_1} f^{(2)}(\mathbf{x}_1, \mathbf{x}_\alpha, \mathbf{v}_1, \mathbf{v}_\alpha, t) \mathbf{dx}_\alpha \mathbf{dv}_\alpha
$$

since all particles of species α **can be considered as identical and a**_{1α} *is the acceleration of particle 1 by a generic particle of the species* α

AS A RESULT THE EQUATION FOR f (1) IS NOT CLOSED

In a plasma the interactions are LONG RANGE INTERACTIONS. Indeed, the inter particle interactions can be divided roughly into two parts:

a) the average force due to all other (distant) particles

b) the force due to the interaction with the nearest neighbor particles closure

The forces due to all other particles (a) does not depend on the exact location of all particles and can be view as "external forces". By neglecting the interactions with the nearby particles (b) we finally get the equation for $f^{(1)}$ *:*

$$
\frac{\partial f^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \mathbf{x}_1} + \frac{q}{m} \left\langle \mathbf{E} + \frac{\mathbf{v}_1 \times \mathbf{B}}{c} \right\rangle \cdot \frac{\partial f^{(1)}}{\partial \mathbf{v}_1} = 0
$$

here \leq **E** $>$ *and* \leq **B** $>$ are the sum of the external and average electromagnetic fields

Neglecting single particle interactions, one make use of the **probability** *f (1) of finding particle 1 at location* **^x***¹ in phase space under the action of the electromagnetic fields generated by the full system* - **mean field theory**

In summary, a plasma is described in a reduced way in terms of the one particle distribution function

 $f^{(1)}(\mathbf{x}_1, \mathbf{v}_1) \, \mathrm{d} \mathbf{x}_1 \mathrm{d} \mathbf{v}_1$

CONTINUUM approach

In other words we define a PLASMA as composed by free charges with **kinetic (thermal) energy much larger than the typical potential energy due to its nearest**

neighbor

? $\Phi_c \iff n_0^{\scriptscriptstyle{\perp}}$ 1/3 *e* 2 $=$ m v_{a,th} 2

where $n_0^{1/3} \sim 1/\lambda$, l = mean particle distance, a=e,i 1/3

 E_k ? $\Phi_C \iff n_0^{1/3}e^2 = m v_{a,th}^2$
where $n_0^{1/3} \sim 1/\lambda$, $l =$ mean particle distance, $a=e, i$
To summarize: we need $\Lambda_D = n_0 \lambda_D^3 \gg \Lambda_D = \sqrt{m v_{th}^2/4 \pi n c^2}$
where $\lambda_D =$ Debye length,
 $\Lambda_D =$ number of particles in a Debye *To summarize: we need* $\left(\widehat{\Lambda}_{\text{D}} = \text{n}_0 \lambda_{\text{D}}^{-3} \implies 1\right)$

 $\lambda_{\rm D} = \sqrt{\text{mv}^2_{\text{ th}}}$ / $4\pi \text{ne}^2$

where

*"***quasi non-correlated***"*

particles

 $\lambda_{\rm D}$ = Debye length,

 $\Lambda_{\rm D}$ = number of particles in a Debye sphere

In such a system, the particle potential falls off much faster than in vacuum: **Debye shielding**

The electric Coulomb potential of each charge

$$
\Phi_c = \frac{q}{4\pi\varepsilon_0 r}
$$

is shielded by the plasma: DEBYE POTENTIAL

long range
interactions system
$$
\Phi_D = \frac{q}{4\pi \varepsilon_0 r} e^{-r/\lambda_D}
$$

Characteristic lengths (frequencies)

IN THIS APPROACH COLLISIONS ARE COMPLETELY NEGLECTED

An example, the **Solar Wind***: no time to reach thermodynamical equilibrium*

often observed in the solar wind

THE ONE PARTICLE D.F. OBEYING LIOUVILLE EQUATION GIVES THE VLASOV EQUATION, THE BASIC EQUATION TO MODEL A COLLISONLESS PLASMA

Here f_a *is the distribution function of a-species, E and B the electromagnetic fields self-consistently generated by the whole system*

> *x and v are phase space coordinates; v is NOT the mean flow used in fluid approaches*

WHAT WE NEED TO KNOW **1 SIMULATE THE PLASMA DYNAM!**

1) A set of equations for the plasma evolution

(FREQUENCY REGIME)

2) A set of equation for the evolution of the electrostatic or electromagnetic fields

3) The boundary conditions

4) The initial conditions

THERE IS NOT A UNIVERSAL MODEL. THE NUMERICAL MODEL AND THE TECHNIQUES TO BE ADOPTED STRONGLY DEPEND ON THE PHYSICAL PROBLEM.

WE ALSO NEED TO IDENTIFY REGIME OF INTEREST (IF POSSIBLE!)

High frequency regime $[\omega \sim \omega_{\text{pe}}]$

•*Vlasov el. + Maxwell [Ion fixed (in general)]*

Intermediate regime $[\omega \sim \Omega_{ci}]$

•*Vlasov ions, fluid electrons + Maxwell*

THE VLASOV – POISSON/MAXWELL SYSTEM MUST BE INTEGRATED 6D+TIME SPACE AND COVERS AN IMPRESSIVE RANGE OF PHYSICAL REGIMES SEPARATED BY MANY ORDER OF MAGNITUDE IN FREQUENCY AND SCALE LENGTHS IT.

NUMERICAL INTEGRATION OF THE VLASOV EQUATION

Eulerian methods "Lagrangian" methods The Vlasov equation is discretized on a grid in phase space and the field equations on a spatial grid A large set, but finite computational "macro-particles" follow the characteristics **"Vlasov codes" Fixed grid "PIC codes" Fixed Maxwell grid; moving macroparticles** Mean field theory **Same physics different numerical approach Here we discuss only Eulerian methods**

The Vlasov equation is basically an advection equation in phase

Liouville's theorem: The phase space volume can be deformed but its density is not changed during the dynamical evolution of the plasma.

It can be considered as a "transport" equation in phase space.

Invariants of the Vlasov equation:

A fundamental feature is that the *d.f*. is subjected to strong topological constraints, provided by the existence of invariants

$$
\frac{d}{dt}\int H(f)\mathrm{d}x\mathrm{d}v = 0
$$
 for any function H

This reduces the (infinite) number degrees of freedom of the system:

the d.f. can be transported and roll up in a complex way in phase space, but *different d.f. iso-lines can never be broken and reconnect*

Transitions from "*unconnected states*" in phase space are forbidden, as for example from a laminar type state (free-streaming) to a vortex type state (particle trapping).

A simple 1D-1V example: the TWO-STREAM instability

This instability can be induced, e.g., by an energetic particle stream injected in a plasma followed by a return current. In practice, we have two counter propagating electron beams at v_b $<<$ *c.* In this *case* the *system evolution is mainly e.s. and can be studied in the 1D-1V limit:*

$$
f_e(x, v, t = 0) = f_{0, e} \left[n_1 e^{-\frac{1}{2}(v_x - v_{01})/v_{th, 1}^2} + n_2 e^{-\frac{1}{2}(v_y - v_{02})/v_{th, 2}^2} \right]
$$

The instability transfer part of the the beam kinetic energy into plasma waves with associate electric fields

First three frames: correct Vlasov evolution

Two-stream instability. Phase space **(x,v) representation**

Long time "non Vlasov" transition between unconnected Vlasov equilibria

Eulerian algorithms are extremely accurate even in the non linear phase. For example they are crucial for the analysis of spectral regions where the energy level of fluctuations is very

Eulerian Vlasov simulations

> But: Eulerian algorithms are highly demanding in terms of CPU and memory!

low.

Eulerian methods

The first step is to discretize the Vlasov equation on a phase space grid and the field equations on a spatial grid and solve the resulting system of finite difference equations by standard HD methods. The main draw back of such a purely "Eulerian" approach is that the time step would be severely limited by CFL type conditions, prohibiting very long simulations, and we will not discuss them any further.*

> *WE NEED A DIFFERENT ALGORITHM WITH RESPECT TO STANDARD FLUID DYNAMICS* (e.g. Spectral Methods, Finite differences, Explicit/Implicit time advancing schemes….)

* Wait for next slide……

The Vlasov equation is an hyperbolic partial differential equation with characteristics given by

For the sake of mathematical simplicity, let consider the electrostatic limit

In Hamiltonian form, using canonical variables $\mathbf{q} = \mathbf{x}$, $\mathbf{p} = m_s \mathbf{v}$:

$$
\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m_a} = \frac{\partial H}{\partial \mathbf{p}}
$$

*d***p** *dt* = **F** ⁼ [−] ∂H $\overline{\partial}$ **q**

Electrostatic limit, $\mathbf{F} = \mathbf{E}$

The phase space trajectories describe thus a "Hamiltonian flow" $T_{\rm t}$

We define $z_0 = (\mathbf{x}_0, \mathbf{v}_0)$ as a phase space point at time $t = 0$ and $z_t = (\mathbf{x}_t, \mathbf{v}_t)$ the point with at time t corresponding to the unique solutions of the characteristics, equations (1):

$$
T_t z_0 = z_t; \quad T_t z_t = z_0
$$

In other words, the "flow" T is REVERSIBLE and preserves the phase space volume element: T_t T_t =*identity ; dz*_{*t*} = *dz*₀

The solution of the Vlasov equation may then be expressed in terms of a propagating operator *P* acting on the *d.f.* If f_0 is the *d.f.* at $t = 0$, then

$$
f(z,t) = P^t f_0 = f_0(T_{-t} z)
$$

This (formal) solution may be termed "LAGRANGIAN" in the sense that the value of $f(z, t)$ at time t depends on the initial coordinates z_0 at t = 0 along the characteristic arriving at the phase space point z at time t.

The Vlasov equation can be rewritten using the Poisson brackets

$$
\frac{\partial f}{\partial t} = [H, f]
$$

By defining the Poisson bracket operator Λ*f* = [H, *f*], the PROPAGATOR P can be written as:

$$
P^t = exp(\Lambda t)
$$

In the free streaming case $(F = 0)$ the characteristic equations have the trivial solution $x = x_0 + vt$, $v = v_0$ and the propagator P^t takes the form of a displacement operator

$$
T^t f(x, y, t) = \exp\left(-t v \frac{\partial}{\partial x}\right) f(x, y, t) = f(x - vt, y, t = 0) = f_0(x_0, v_0)
$$

Except for the free streaming case one does not know explicit expressions of the propagator *P***^t even for very simple physical problems**. As a consequence, the theoretical study of the Vlasov–Poisson/Maxwell system of equations is based today on *large-scale numerical simulations*.

MAIN NUMERICAL SCHEMES

1. Fully Lagrangian schemes

Grid-free Lagrangian particle simulation methods for collisionless plasmas are the only truly Lagrangian ones. In these simulation methods the force exerted on a given particle, by the other ones is explicitly calculated with some methods.

The computational cost is very high typically $O(Np \log Np)$.

Applied only to low dimensial (1D-1V) test cases.

2. Particle in Cell methods (the most used in plasma physics)

They use "**Macro-particles**" **which are not particles but a representation of the d.f.**

- a) Solve continuum equations on an Eulerian grid (Poisson or Maxwell equations, electron equations for the hybrid case)
- b) Track the individual particles by solving the equations of motion;
- c) Couple the Eulerian to the Lagrangian framework by interpolating the fields to the particle positions, $E(x_p)$, $B(x_p)$;
- d) Couple the Lagrangian to Eulerian framework, by evaluating the values of the electric charge and current densities at the spatial grid points **x**^j

Although the solution of the dynamical equations in the second step introduces some error and noise, the "noise in particle simulations" is predominantly associated with the fourth step where low-order moments of the distribution function are calculated to find the source terms for Poisson's or Ampere's equations

3. Semi-Lagrangian methods

Instead of tracking numerical particles along the characteristics of the Vlasov equations, these methods relie on a discretization of the phase space but following the characteristic curves at each time step. For this reason they are known under the generic appellation of "Semi Lagrangian" schemes, although the terminology is not well settled.

IN THE FOLLOWING WE WILL FOCUS ON THESE SCHEMES

GRID DISCRETIZATION

Assume the flow T t *is known;* **the problem is then to obtain ^a discrete formulation** *to integrate the Vlasov equation*

The *d.f.* $\{f_m(t)\}\$ is discretized on a fixed Eulerian grid $\{z_m\}$

Since the *d.f.* f is conserved along the flow, the *d.f.* at the new time step $\{f_m(t+\Delta t)\}$ is calculated by following backwards the characteristic during the time step Δt and by interpolating the values of ${f_m(t)}$ at the origin of the characteristics.

Main algorithm aspect: we need to interpolate the *d.f.*

The trajectory of each Lagrangian point \mathbb{Z}_m at t_n+ Δt is integrated backwards in time to find the "departure point" Z_{m}^{d} at the earlier time t_{n} . Then:

$$
f(\mathbf{Z}_{\mathrm{m}}, t_{\mathrm{n}} + \Delta t) = f(\mathbf{Z}_{\mathrm{m}}^{\mathrm{d}}, t_{\mathrm{n}}) = f(\mathbf{Z}_{\mathrm{m}} - \mathbf{D}\mathbf{Z}_{\mathrm{m}}, t_{\mathrm{n}})
$$

The *"displacement vector*" $DZ_m = Z_m(t_n + \Delta t) - Z_{m}^d(t_n)$ is obtained through some numerical approximation of the equations of motion usually at second order in time

Then an interpolation is used to obtain the values $f(Z^d_m, t_n)$ of the distribution function at the departure points from the known values on the mesh grid $\{f_m(t_n)\}\.$

SUMMARY

(a) determination of the departure points Z_{m}^{d}

(b) an interpolation of the distribution function to the departure points

INTERPOLATION TECHNIQUES

We focus here only on Finite elements discretizations technique $(C_m = m \text{ cell})$

$$
\langle f_m \rangle(t) = \frac{1}{V_m} \int_{C_m} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v}
$$

From the Vlasov equation one may obtain an exact equation relating the time variations of $\leq f_m$ to the fluxes crossing the cell boundaries. These fluxes depend on the value of the distribution function *f* at the cell boundaries but not on the cell averages $\{\leq f_m>\}$

A critical step is to express the fluxes at the cell interfaces in terms of the cell averages, a procedure implying an interpolation; once this is done, one obtains an explicit discretization scheme [Finite Volume Method] a mass conservative, robust and computationally "cheap" method for the discretization of conservation laws.

Mathematically, the finite element discretization of a function $f(Z)$ is given by a linear combination of a finite number S of basis functions $\xi_{\text{ms}}(\mathbf{Z})$, $s = 0, \dots$, S for each cell (projection of *f* on a finite dimensional functional space having a finite complete basis χ_{ms} and the corresponding dual basis $\xi_{\rm ms}$:

$$
f(\mathbf{Z}) = \sum_{m,s} a_{m,s}(t) \chi_{m,s}(\mathbf{Z}) f(\mathbf{Z}); \quad a_{ms} = \int \xi_{m,s}(\mathbf{Z}) f(\mathbf{Z}) d\mathbf{Z}
$$

$$
\int \xi_{m',s'}(\mathbf{Z}) \chi_{m,s}(\mathbf{Z}) d\mathbf{Z} = \delta_{m,m'} \delta_{s,s'}
$$

The basis functions may be considered as providing a low order interpolation inside each cell; [e.g. a piecewise linear basis functions leads to a first order interpolation].

The solution of the Vlasov equation can be expressed in terms of a propagator describing the flow in phase space:

$$
f(\mathbf{Z}, t_n + \Delta t) = P^{\Delta t} f(t) = f(T^{(-\Delta t)} \mathbf{Z}, t)
$$

If the coefficients $a_{m,s}(t)$ are known at time t, then $f(\mathbf{Z}, t + \Delta t)$ at a later time can be written in terms of these known coefficients:

$$
f(\mathbf{Z}, t + \Delta t) = \sum_{m,s} a_{m,s}(t) \chi_{m,s}(\mathbf{T}^{(-\Delta t)}\mathbf{Z}, t)
$$

By projecting onto the finite element dual basis $\xi_{\text{m.s}}$, it is possible to calculate explicitly the coefficients $a_{m,s}(t_n+\Delta t)$

These method, known as "discontinuous Galerkin method", relies on the projection on a function space spanned by a finite basis of functions, without requiring any condition on the regularity of f and its derivatives at the cell boundaries

TIME ADVANCING ALGORITHM (*electrostatic limit*)

approximation of the flow evolution operator T

the Vlasov equation can be cast in the form

$$
\frac{\partial f}{\partial t} = [H, f] = \Lambda f
$$

Then, the propagation operator becomes $P^t = e^{\Lambda t}$

Free streaming $(F_L = 0)$, trivial solution for the characteristic equations

$$
x = x_0 + vt
$$
; $v = v_0$ $Pt f(x, v, t) = f(x-vt, v, t=0) = f(x_0, v_0)$

¹ C. G. Cheng and G. Knorr, J. Comput. Phys. **22**, 330 (1976)

SPLITTING SCHEME

We split the Hamiltonian in two parts corresponding to the two separate advections in space and velocity

$$
\frac{\partial f}{\partial t} = [H_1, f] = -v \frac{\partial f}{\partial x}; \qquad \frac{\partial f}{\partial t} = [H_2, f] = E \frac{\partial f}{\partial v} \qquad H = H_1 + H_2
$$

H₁ = v²/ 2 ; H₂ = -\Phi (x)

the corresponding x and v propagators are explicitly known:

$$
f(x, v, t) = f_0(x - vt, v) \equiv P_x^t[f_0]; \ \ P_x^t = \exp(\Lambda_1 t)
$$

$$
f(x, v, t) = f_0(x, v + Et) \equiv P_x^t[f_0]; \ \ P_y^t = \exp(\Lambda_2 t)
$$

Since the operators Λ_1 and Λ_2 do not commute, $P^{\Delta t}$ does not reduce to the product $P_x^{\Delta t} \times P_y^{\Delta t}$ but instead may be given by a series expansion in Δt involving ordered products of $P_x^{\Delta t/n} \times P_y^{\Delta t/m}$, with n, m integers

It can be shown that

$$
e^{\Lambda t} = \lim_{N \to \infty} \left[\exp\left(\frac{\Lambda_2 t}{2N}\right) \exp\left(\frac{\Lambda_1 t}{N}\right) \exp\left(\frac{\Lambda_2 t}{2N}\right) \right]^N
$$

We then make use of a "Magnus expansion" at second order:

$$
e^{\Lambda \tau} = \exp\left(\frac{\Lambda_2 \tau}{2}\right) \exp\left(\Lambda_1 \tau \right) \exp\left(\frac{\Lambda_2 \tau}{2}\right) + O\left(\tau^3\right)
$$

The full Vlasov propagator can be therefore expressed in terms of the two space and velocity propagators

$$
P^{d\tau}\left[f(x,v)\right] = \left\{P_x^{dt/2} + P_v^{dt} + P_x^{dt/2}\right\}\left[f(x,v)\right] + O\left(dt^3\right)
$$

which would be exact if the propagators would commute

SPLITTING ALGORITHM, e.s. CASE

 $F(x, v, t+dt) = P^{dt} [f(x,v)] = {P_x^{dt/2} P_v^{dt} P_x^{dt/2} } [f(x,v)] +$ *O(dt3)*

> *1)* $f^*(x, v) = P_x^{\text{dt}/2} [f(x, v, t)]$ 3) $f^{\#}(x,v) = P_v^{\mathrm{dt}} [f^*(x,v)]$ $\nabla^2 \phi = n_i$ $-\int \int f^*$ 2) $\nabla^2 \phi = n_i - \int f^*(x, v) dv$

4) $f(x, v, t+dt) = P_x^{\text{dt}/2} [f^{\#}(x, v)]$

SPLITTING ALGORITHM, e.m. CASE

For the sake of simplicity, we consider ^a one directional, uniform magnetic field along the z-axis and the 1D-2V limit (x, vx, vy). Lorentz force: $E_{\!x\!}$ +v_y $B_{\!z}$; $E_{\!y\!}$ +v_x $B_{\!z}$

We split the full Vlasov propagator into ^a sum of ^a propagation in space and ^a propagation in velocity, this last corresponding to ^a translation plus ^a semi-rotation.

$$
\mathfrak{R}_{vx}^{dt} f(x, v_x, v_y) = f(x, v_x + [E_x + v_y B_z] dt, v_y)
$$

$$
\mathfrak{R}_{vy}^{dt} f(x, v_x, v_y) = f(x, v_x, v_y + [E_y - v_x B_z] dt)
$$

The 1D-2V full advancement of the Vlasov equation is now obtained as:

 $P^{dt}[f(x, v_x, v_y)] = P_x^{\frac{dt}{2}} \left[\Re^{dt/2}_{vx} \Re^{dt}_{vy} \Re^{dt/2}_{vx} \right]$ $\overline{}$ $\int P_x^{dt/2} [f(x, v_x, v_y)] + O(dt^3)$

* Mangeney & Califano, J. Comp. Phys. (2002)

The scheme thus reduces to follow backwards the characteristic during the *time step dt and interpolating the values of* $\{f_m(t_n)\}$ *at the origin of the* **characteristics.**

Main interpolation methods

Cubic splines

A piecewise third-order polynomial function passing

through ^a set of control points. The second derivative

of each polynomial is commonly set to zero at the endpoints (boundary condition)

> *1. Charge not conserved 2. Non Local Stencil*

[*3-diag. matrix to be inverted***]**

Van Leer scheme II or III order, M=1,2

$$
\bar{f}_i(t + \Delta t) = \sum_{j = -(M+1)}^{M} A_j(\delta) \bar{f}_{i+j}(t), \quad v > 0
$$

$$
\bar{f}_i(t + \Delta t) = \sum_{j = -M}^{M+1} A_j(\delta) \bar{f}_{i-j}(t), \quad v < 0
$$

δ=vΔt/Δx [i.e. it depends on the "signed" CFL number! But much more stable than using standard HD methods.

1. Charge conserved 2. Local Stencil [*well suited for parallel computations*]

The upwind schemes attempt to discretize hyperbolic partial differential equations by using differencing biased in the direction determined by the sign of the characteristic speeds

RESEARCH APPLICATION EXAMPLES

Vlasov simulations, running on super computers….

*The Hybrid Model**

Kinetic ions (solve Vlasov equation)

Quasi-neutrality is satisfied

Displacement current is negligible

The plasma is weakly magnetized

$$
\ell \Box \quad \lambda_{D_e} \, ; \, \omega = \, \omega_{pe}
$$

$$
\Omega_{ce} = \, \omega_{pe}
$$

Electron are considered as a fluid but with mass: $d_e = \sqrt{m_e / m_i}$

The equations

$$
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \mathbf{F}_{ext} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0
$$

External forcing

Vlasov (ions)

$$
\mathbf{E} = -(\mathbf{u} \times \mathbf{B}) + \frac{1}{n} (\mathbf{j} \times \mathbf{B}) - \frac{1}{n} \nabla P_e + d_e^2 [\Delta \mathbf{E} + \dots]
$$

Ohm law - hybrid (electron response)

$$
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad Faraday
$$

Characteristic quantities (normalization on ion time scale)

 $\mathbf{j} = \nabla \times \mathbf{B}$ low frequencies $n_{\rm e} \approx n_{\rm i} \approx n$ Quasi neutrality $P_e = n^{\gamma}$ State equation $=$ n^{γ} en_{e} **u**_e $=$ e n_{i} **u**_i - **J**

 $\wedge_i = eB/m_ic$ $d_i = c/\omega_{pi} = v_A / \sqrt{i}$

$$
d_e^2 = m_e \mathbin{/} m_{ii}
$$

* Valentini & Califano, J. Comp. Phys. (2007)

The numerical algorithm consists in coupling the splitting method by Mangeney & Califano (JCP 2002) and the current advance method (CAM) introduced by Matthews in 1994 for PIC Hybrid simulations*

The Vlasov-CAM method (see Valentini & Califano, JCP 2007) provides the second order in time numerical solution for the advancement of electric and magnetic fields, while the splitting method is a second order scheme in time for the advance of the particle distribution function in phase space.

The CAM method is introduced because the standard integration of the Maxwell equations, at II order, would be to heavy computationally (remember that n and **j** are integrals of f(**x**, **v**))

^{*} *A.P. Matthews*, J. Comp. Phys. (1994)

2D-3V Vlasov hybrid simulations of "perpendicular turbulence"

BG/Q (Cineca, IT) Hydra (RZG, DE)

β ≈ 0.2, 1, 5; T^e ≈ Tⁱ Maxwellian distribution Initial conditions

Forcing on m = 1, 2 compressible or incompressible

 N_x , N_y = 256, 512, 1024 Nv_x , Nv_y , $Nv_y = 71^3$ (51³) $1024 \le N_{\text{procs}} \le 8192$

SETUP OF 2D3V HVM SIMULATIONS

two dimensions in physical space and three dimensions in velocity space

 $N_x = 1024$ $N_y = 1024$ $N_{Vx} = 51$ $N_{Vy} = 51$ $N_{Vz} = 51$ *Phase Space Discretization Periodic in physical space* $f(|v| > v_{\text{max}}) = 0$ $v_{\text{max}} = \pm 5v_{th}$ *boundary conditions*

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25-8-2014 7th ITER International School 56 **k**
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25-8-2014 7th ITER International *The initial Maxwellian equilibrium is perturbed by a 2D spectrum of fluctuations for magnetic and proton velocity field. The energy is injected with random phases. Neither density disturbances nor parallel variances are imposed at t=0.*

$\rm t \sim 100 \, \Omega_{ci}^{-1}$ $β \sim 1$, initial phase (compressible forcing)

The magnetic and electric energy spectrum. Only MHD scales are excited

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β ~ 1 case, second phase current sheets generation

The perp. current J_z *and electric field* E_z *in the (x,y) plane*

We have changed the numerical resolution to check the numerical correctness. The width of the CS is regulated by the numerical dissipation of the algorithm

we observe magnetic islands chains

The density n_p *and perp. magnetic field* B_z *in the physical* (x,y)

formation of magnetic islands chains

We observe strong (anti) correlation between np and B^z (inside) outside the current sheets

The magnetic and electric energy spectrum after CS formations

RECONNECTION EVENTS IN TURBULENCE

Reconnection also observed in Alfven waves decaying turbulence We observe vortices, islands, current sheets, but…also anisotropy in velocity space (see next slide)

Thickness ≈ few proton skin depths

VELOCITY DISTRIBUTIONS IN TURBULENCE

Local magnetic fields \hat{e}_1 \hat{e} \hat{e}_3 $\hat{\mathbf{B}}$ nê, \hat{e}

Anisotropy with respect to local magnetic field can be either >1 or <1

> [Servidio et al., 2012; Perrone et al., 2013]

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The ion distribution functions are strongly affected by turbulence, resembling an elongated potato-like structure

 -0.06

 -0.05 -0.04 -0.03 -0.02 -0.01 -0.0

WE CAN MEASURE TEMPERATURE ANISOTROPY!

The ion distribution functions exhibit strong deformations in velocity space

How to properly measure these distortions?

Assuming f as an ellipsoid

Stress tensor

$$
A_{ij}(\mathbf{x}) = \frac{1}{n} \int \left(v_i - \left\langle v_i \right\rangle\right) \left(v_j - \left\langle v_j \right\rangle\right) f d^3 v
$$

e ˆ1

Eigenvalues (temperatures) Eigenvectors

e ˆ 3 $l_1 > l_2 > l_3$ *e* ˆ2

Note: for a Maxwellian $\lambda_1 = \lambda_2 = \lambda_3 = 1$

(Maximum) Temperature anisotropy = λ¹ /λ3

HVM RESULTS VS SOLAR WIND DATA

Trying to reproduce the solar-wind anisotropy plot

We considered an ensemble of simulations in different regions of the parameter space and evaluated the temperature anisotropy with respect to the local magnetic field

NICE AGREEMENT

Firehose instability - benchmarks

$$
p_{\rm P} - p_{\perp} > B^2 / 4\pi
$$

we initialize using a bi-Maxwellian

$$
f(\mathbf{v}) = \frac{n_0}{\pi^{3/2} \beta_P^{1/2} \beta_\perp} \exp\left(-\frac{v_{\rm p}^2}{\beta_{\rm p}} - \frac{v_{\perp}^2}{\beta_\perp}\right)
$$

instability parameter ($\beta \sim 100$ *)*

 $\Lambda = (\beta_P - \beta_{\perp} - 2)/\beta$; 0.01

Firehose instability - benchmarks

parameters: $\beta = 95; \ \beta_{\rm p} = 97; \ \beta_{\perp} = 94; \ \ \ \text{space box: L}_{\rm x} = 2\pi \ \ \text{*} \ 500$

2D-3V firehose investigation

 \rightarrow co-existing parallel and oblique firehose instabilities

code parameters: space box: 768x768, velocity box: 81x81x81 Lx = Ly = 2π * 76.394 (Δx = 0.625), $v_x = v_y = v_z = [-5.0, 5.0]v_{th}$ (Δv = 0.123)

THANK YOU FOR YOUR ATTENTION !

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