IDEAL MHD STABILITY THEORY…

… and tokamak operational limits

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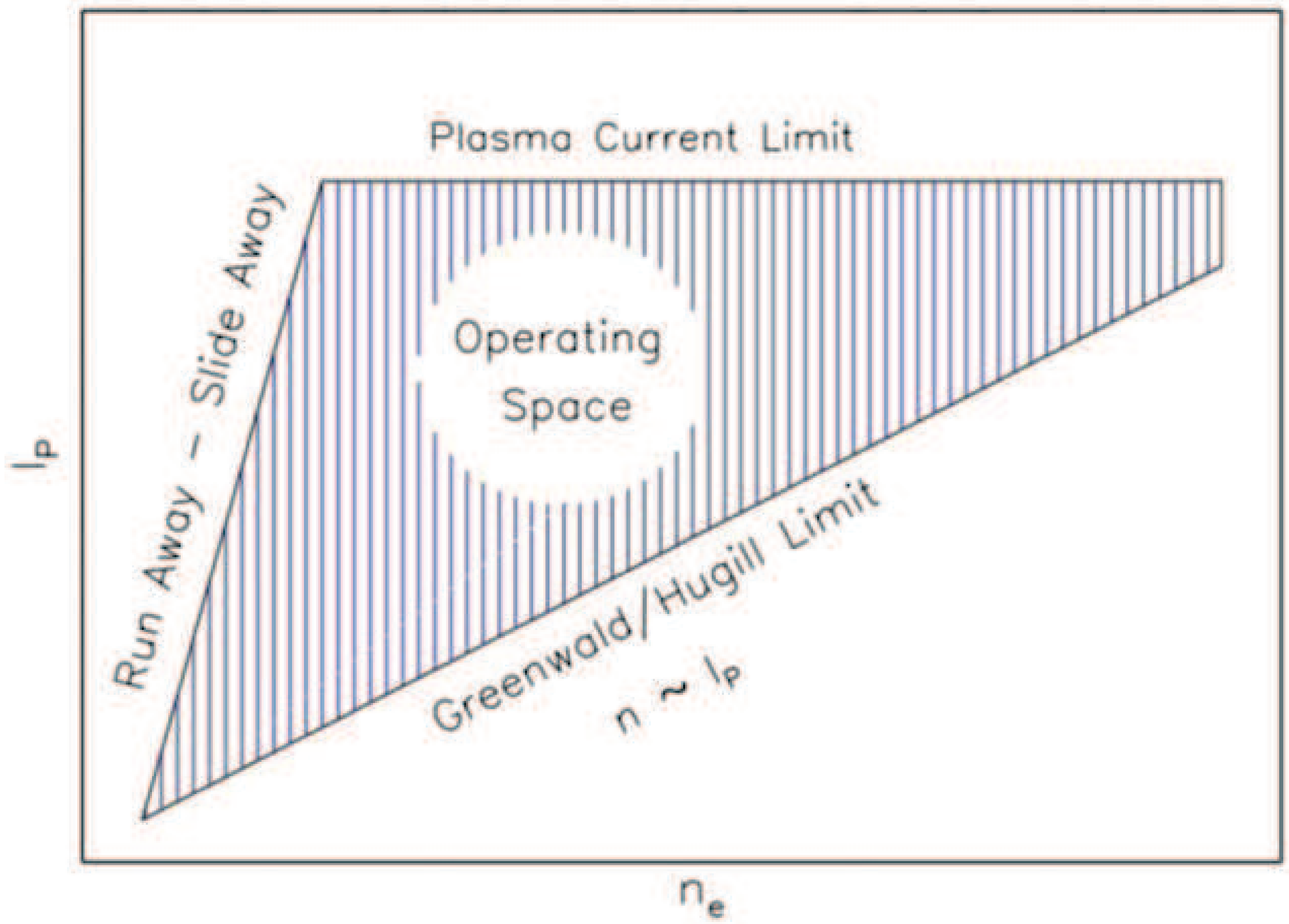
Content of this lecture

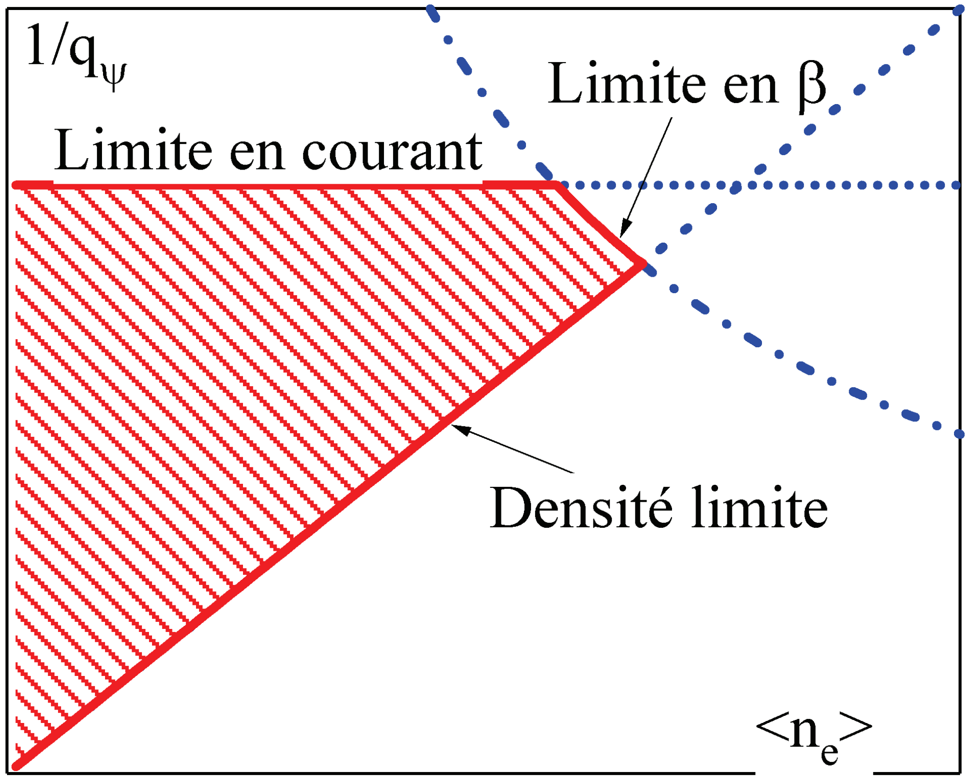
- Reduced ideal MHD model
- Current driven instabilities $\longrightarrow$ current limit
- Pressure driven instabilities $\longrightarrow$ beta limit

How does the current limit appear in an MHD model?



[Greenwald 2002]

Additionally: ideal MHD beta limit



Constructing the reduced ideal MHD model I

- ideal MHD equations:

  charge balance: $\nabla \cdot \mathbf{j} = 0$ Ohm's law: $\mathbf{E} + \mathbf{u} \times \mathbf{B} = (\eta \mathbf{j})$

  Ampère's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ press. bal.: $\partial_t p + \nabla \cdot (p\mathbf{u}) + \frac{2}{3} p \nabla \cdot \mathbf{u} = 0$

- reduced MHD approximation: "strong" external magnetic field $\mathbf{B}_0$

  – perpendicular motion is described by drifts

$$\mathbf{u}_{e,i_\perp} = \frac{\mathbf{B} \times \nabla \phi}{B^2} + \frac{\mathbf{B} \times \nabla p_{e,i}}{neB^2} + \frac{m_i \mathbf{B}}{eB^2} \times \frac{\mathrm{d}\mathbf{u}_E}{\mathrm{d}t}, \quad \mathbf{j}_\perp = en\left(\mathbf{u}_{i_\perp} - \mathbf{u}_{e_\perp}\right)$$

ExB drift $\mathbf{u}_E$ diamagnetic drift polarization drift

  – electromagnetic fields are described by potentials

$$\mathbf{E} = \left(\partial_t \psi - \nabla_\parallel \phi\right) \frac{\mathbf{B}_0}{B_0} - \nabla_\perp \phi, \quad \mathbf{B} = \mathbf{B}_0 + \frac{\mathbf{B}_0}{B_0} \times \nabla \psi$$

Constructing the reduced ideal MHD model II

- charge balance: $\nabla \cdot \mathbf{j} = \nabla \cdot \left(\frac{\mathbf{B} \times \nabla p}{B^2} + \frac{n m_i}{B^2} \mathbf{B} \times \frac{\mathrm{d}\mathbf{u}_E}{\mathrm{d}t} \right) + \nabla_{\parallel} j_{\parallel} = 0$
  - with $\nabla \cdot \frac{\mathbf{B} \times \nabla p}{B^2} \approx 2 \left(\frac{\mathbf{B}}{B^2} \times \kappa \right) \cdot \nabla p$ where $\kappa = \left(\hat{\mathbf{b}} \cdot \nabla \right) \hat{\mathbf{b}}$

    magn. curvature
  - and $\nabla \cdot \left(\mathbf{B} \times \frac{\mathrm{d}\mathbf{u}_E}{\mathrm{d}t} \right) \approx -\nabla \cdot \frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\perp} \phi \approx -\frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\perp}^2 \phi$
  - gives $$\boxed{\frac{n m_i}{B^2} \frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\perp}^2 \phi = \nabla_{\parallel} j_{\parallel} + 2 \left(\frac{\mathbf{B}}{B^2} \times \kappa \right) \cdot \nabla p}$$
- modelling of current driven instab.: curvature+ polarization current not needed

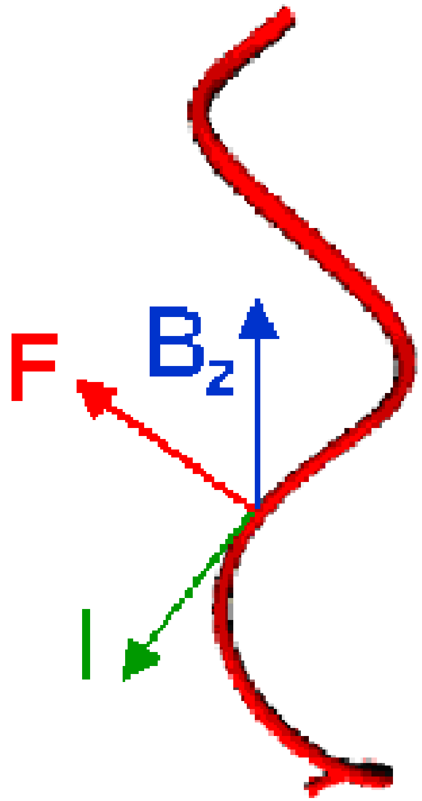
$\rightarrow$ $\boxed{\nabla_{\parallel} j_{\parallel} = 0}$ [but for pressure driven instabilites $\rightarrow$ see later]

Modelling current driven instabilities

- magnetic field: $\mathbf{B} = \mathbf{B}_0 + \frac{\mathbf{B}_0}{B_0} \times \nabla\psi_{\text{eq}} + \frac{\mathbf{B}_0}{B_0} \times \nabla\psi$ (equilibrium + perturbation)
- Ampère's law:
  - equilibrium: $\mu_0 \mathbf{j}_{\text{eq}} = \nabla \times \left(\mathbf{B}_0 + \frac{\mathbf{B}_0}{B_0} \times \nabla\psi_{\text{eq}} \right)$
  - perturbation: $\mu_0 \tilde{j}_{\|} \approx \frac{\mathbf{B}_0}{B_0} \cdot \left[\nabla \times \left(\frac{\mathbf{B}_0}{B_0} \times \nabla\psi \right) \right] \approx \nabla_\perp^2 \psi$
- charge balance: $\nabla_\| j_\| = 0$

$$\rightarrow \boxed{\left(\mathbf{B}_0 + \frac{\mathbf{B}_0}{B_0} \times \nabla\psi_{\text{eq}} \right) \cdot \nabla\nabla_\perp^2 \psi + \mu_0 \left(\frac{\mathbf{B}_0}{B_0} \times \nabla\psi \right) \cdot \nabla j_{\text{eq}\|} = 0}$$

Basic view of a kink instability

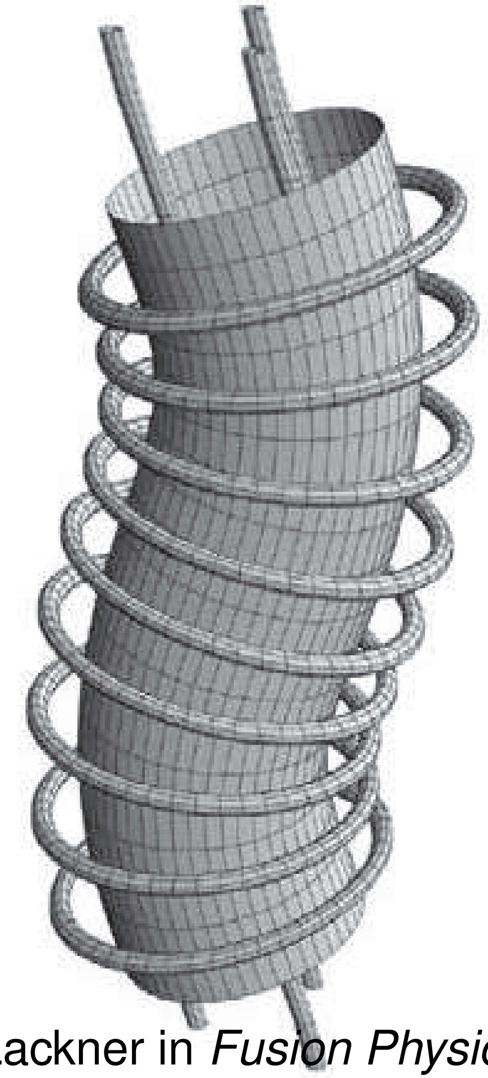


[Garbet, cours Master SFP, AMU]

[on blackboard]

- straight wire carrying a current $\mathbf{I}_0 = I_0 \mathbf{e}_z$
- is placed in a uniform magnetic field parallel to the direction of the wire, $\mathbf{B} = B\mathbf{e}_z$
- is deformed helically:
  $\mathbf{x}(z) = \xi_r \cos(kz)\, \mathbf{e}_x + \xi_r \sin(kz)\, \mathbf{e}_y + z\mathbf{e}_z$
- current $\mathbf{I}$ flowing in the twisted wire: ...
- Lorentz force $\mathbf{I} \times \mathbf{B}$ acting on twisted wire: ...
- equation of motion $\rightarrow$ growth rate ...

Kink instability in cylindrical geometry



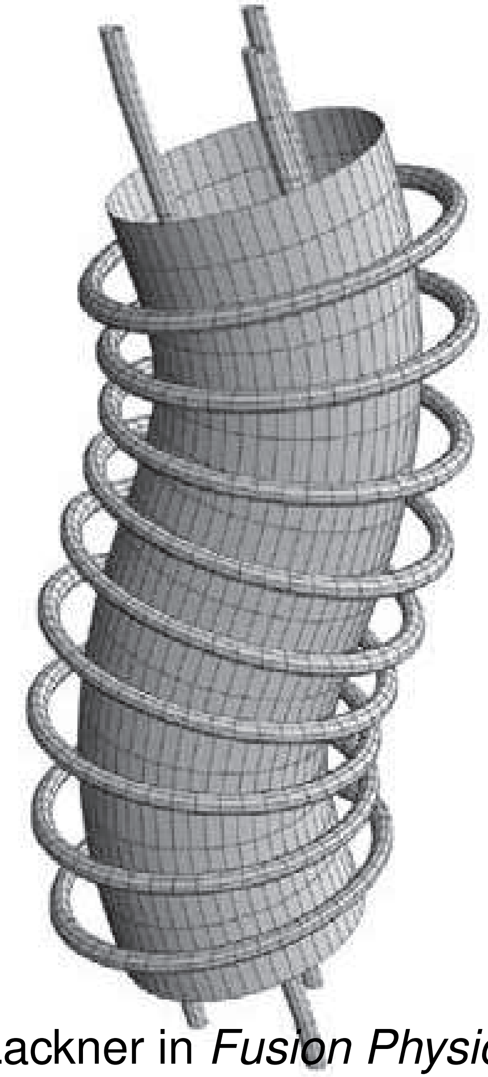
[Lackner in *Fusion Physics*]

- equilibrium: $\mathbf{B}_0 = B_0 \mathbf{e}_z\,, \quad \mathbf{e}_z \times \nabla\psi_{\mathrm{eq}}(r) = \frac{\mathrm{d}\psi_{\mathrm{eq}}}{\mathrm{d}r}\mathbf{e}_y$
- safety factor: $\frac{1}{q(r)} = \frac{R_0}{rB_0}\frac{\mathrm{d}\psi_{\mathrm{eq}}}{\mathrm{d}r}$
- charge balance:

$$B_0\left(\mathbf{e}_z + \frac{r}{R_0 q}\mathbf{e}_y\right)\cdot\nabla\nabla_\perp^2\psi + \mu_0(\mathbf{e}_z \times \nabla\psi)\cdot\nabla j_{\mathrm{eq}\parallel} = 0$$

- perturbation: $\psi = \tilde{\psi}(r)\exp\left(\mathrm{i}m\theta - \mathrm{i}n\frac{z}{R_0}\right)$
  - resonant at $r = r_s$ with $q(r_s) = \frac{m}{n}$
  - marginally stable!
  - Does such a solution exist?

Kink instability in cylindrical geometry



[Lackner in *Fusion Physics*]

$$B_0\left(\mathbf{e}_z+\frac{r}{R_0q}\mathbf{e}_y\right)\cdot\nabla\nabla_\perp^2\psi+\mu_0\left(\mathbf{e}_z\times\nabla\psi\right)\cdot\nabla j_{\mathrm{eq}_\parallel}=0$$

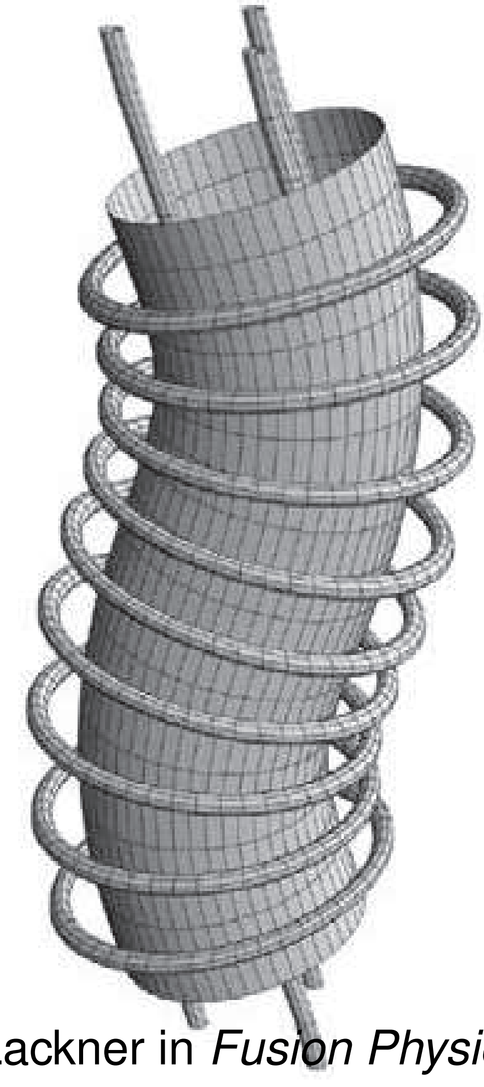
$$\rightarrow\quad -\frac{B_0}{R_0}\left(n-\frac{m}{q}\right)\nabla_\perp^2\psi-\mu_0\frac{m}{r}\frac{\mathrm{d}j_{\mathrm{eq}_\parallel}}{\mathrm{d}r}\psi=0$$

$$\rightarrow\quad \left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)-\frac{m^2}{r^2}+\frac{\mu_0}{\frac{n}{m}-\frac{1}{q}}\frac{R_0}{rB_0}\frac{\mathrm{d}j_{\mathrm{eq}_\parallel}}{\mathrm{d}r}\right]\tilde{\psi}=0$$

exact solution for $m=1$, $n=1$:

$$\tilde{\psi}(r)=\begin{cases} r\left(1-\frac{1}{q(r)}\right) & 0<r<r_s \\ 0 & r_s<r<a \end{cases}$$

Kink instability in cylindrical geometry



[Lackner in *Fusion Physics*]

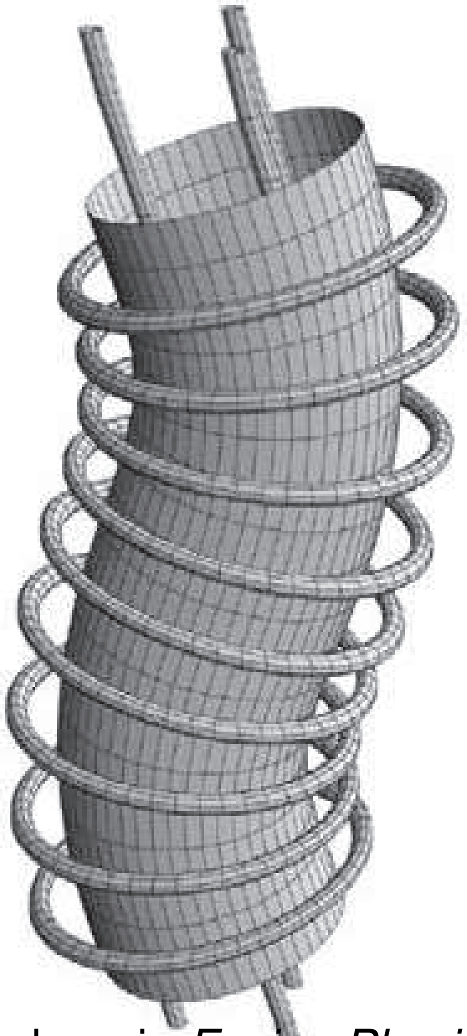
- exact solution for $m = 1, n = 1$ unstable if $q(0) < 1$ (internal kink)
- for $m, n \neq 1$, for a step current profile

$$j_{eq\|}(r) = \begin{cases} -j_0' r_0 & 0 < r < r_0 \\ 0 & r_0 < r < a \end{cases}$$

continuous solutions can be constructed

$$\tilde{\psi}(r) = \begin{cases} \psi_0 \left(\frac{r}{r_0}\right)^m & 0 < r < r_0 \\ \psi_0 \frac{\left(\frac{r}{r_s}\right)^m - \left(\frac{r}{r_s}\right)^{-m}}{\left(\frac{r_0}{r_s}\right)^m - \left(\frac{r_0}{r_s}\right)^{-m}} & r_0 < r < r_s \\ 0 & r_s < r < a \end{cases}$$

Kink instability in cylindrical geometry



[Lackner in *Fusion Physics*]

discontinuity of current profile

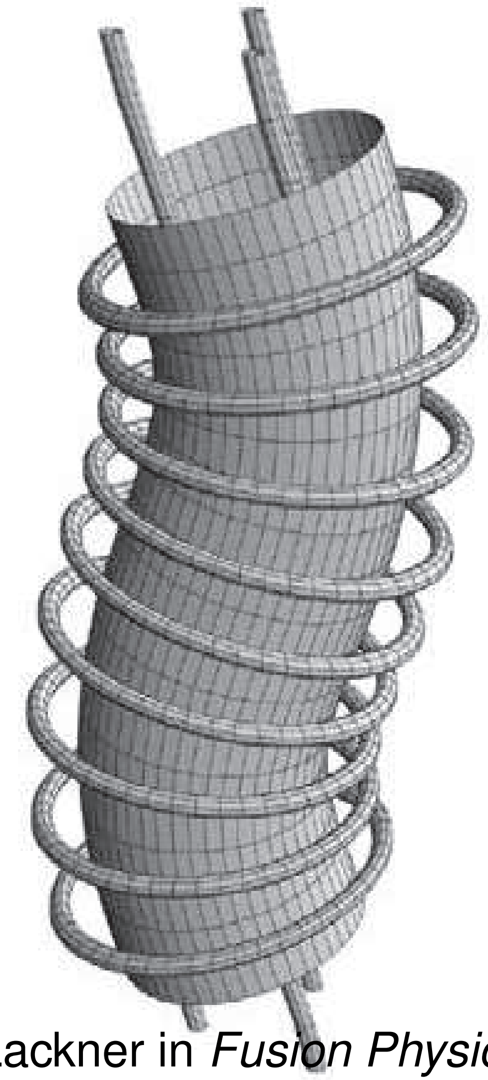
$$\frac{\mathrm{d}\, j_{\mathrm{eq}\|}}{\mathrm{d}r} = j_0' r_0 \delta(r - r_0) \quad (j_0' < 0)$$

implies jump in slope of $\tilde{\psi}$; solution exists if

$$-\frac{2m}{r_0}\frac{1}{1-\left(\frac{r_0}{r_s}\right)^{2m}} = \frac{\mu_0}{\frac{1}{q}-\frac{n}{m}}\frac{R_0}{B_0}j_0'$$

i.e. for a negative current gradient located inside the resonant surface ($r_0 < r_s$) (internal kink).

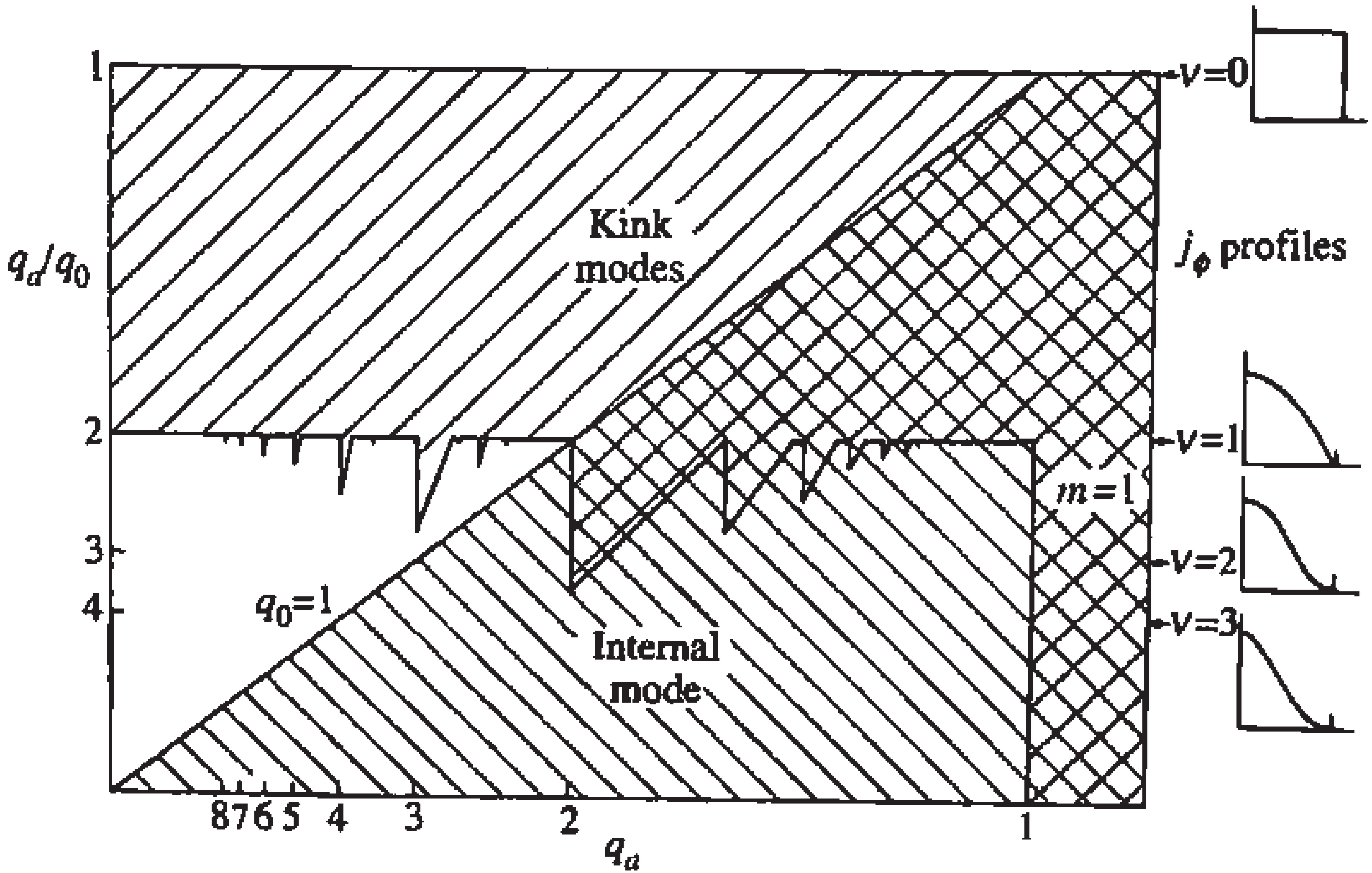
Kink instability in cylindrical geometry



[Lackner in *Fusion Physics*]

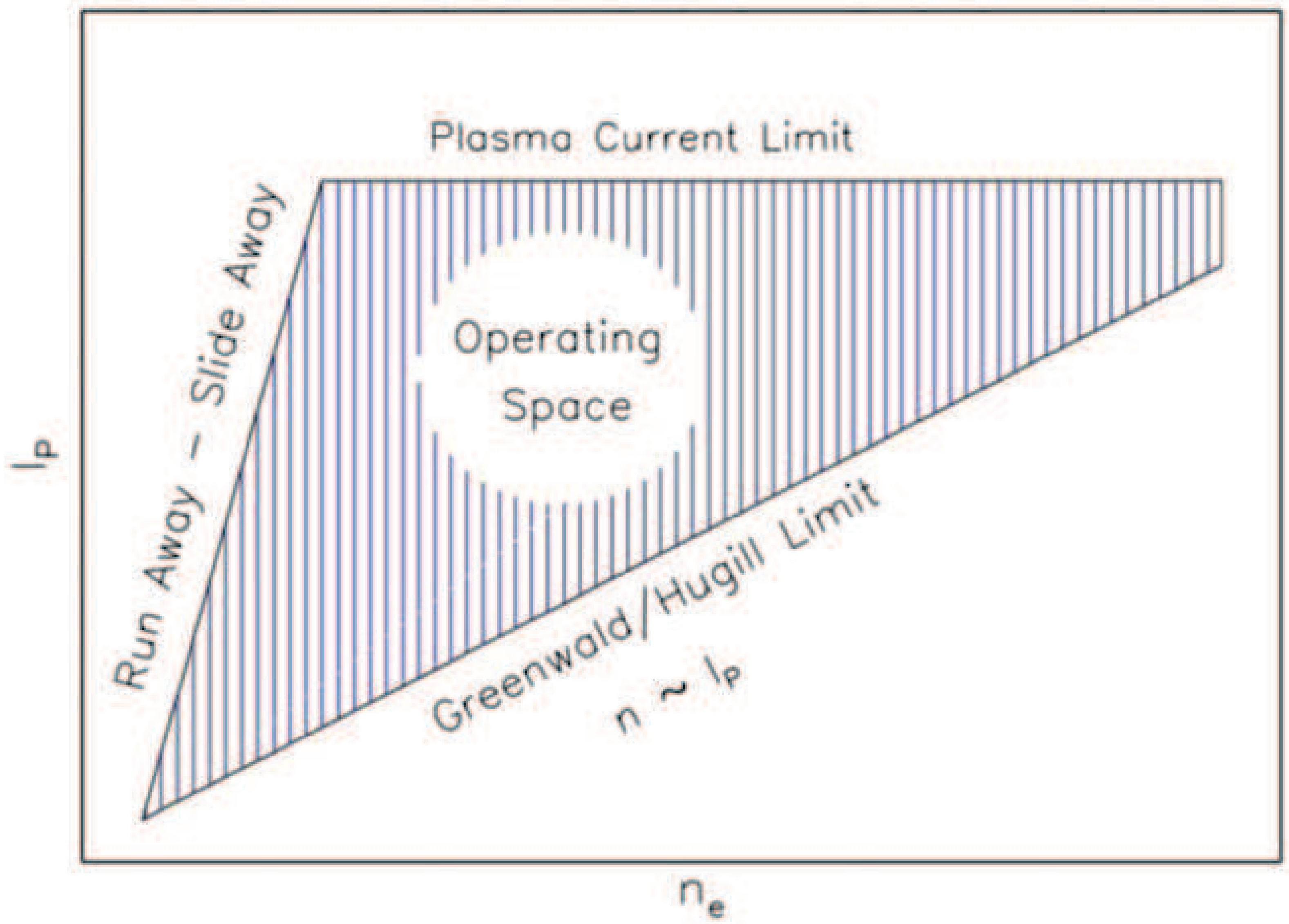
- similar analysis is possible for external kinks (wall pushed to ∞)
- $m = 1$, $n = 1$ external kink is unstable if $q(a) < 1$ (Kruskal Shafranov limit)
- $m, n \neq 1$ external kink modes imply even stronger limits $\rightarrow$ see e.g. diagram by Wesson

Internal + external kink limit current to $q_a \gtrsim 2-3$



[Wesson, Tokamaks]

Operational space



[Greenwald 2002]

Pressure driven modes: back to MHD model

- ideal MHD equations:

  charge balance: $\nabla \cdot \mathbf{j} = 0$ Ohm's law: $\mathbf{E} + \mathbf{u} \times \mathbf{B} = (\eta \mathbf{j})$

  Ampère's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ press. bal.: $\partial_t p + \nabla \cdot (p\mathbf{u}) + \frac{2}{3} p \nabla \cdot \mathbf{u} = 0$

- reduced MHD approximation:
  - charge balance: $$\frac{n m_i}{B^2} (\partial_t + \mathbf{u}_E \cdot \nabla) \nabla_\perp^2 \phi = \nabla_\parallel j_\parallel + 2 \left(\frac{\mathbf{B}}{B^2} \times \kappa \right) \cdot \nabla p$$
  - Ohm's law: $$\partial_t \psi = \nabla_\parallel \phi + \left(\frac{\eta}{\mu_0} \nabla^2 \psi \right)$$
  - Pressure balance: $$(\partial_t + \mathbf{u}_E \cdot \nabla) p = \frac{10}{3} p_0 \left(\frac{\mathbf{B}}{B^2} \times \kappa \right) \cdot \nabla \phi$$

Basic view of an interchange instability

- simple "slab" geometry: $\mathbf{B}_0 = B_0 \mathbf{e}_z$, $\boldsymbol{\kappa} = \kappa \hat{e}_x$
- equilibrium: $\phi_{\text{eq}} = 0$, $\psi_{\text{eq}} = 0$, $p_{\text{eq}} = p_0' \, (x - a)$
- perturbation: $\tilde{\phi}, \tilde{\psi}, \tilde{p} \sim \exp\left(\mathrm{i}k_x x + \mathrm{i}k_y y + \mathrm{i}k_\parallel z + \gamma t\right)$
- linear disperson relation: [blackboard]

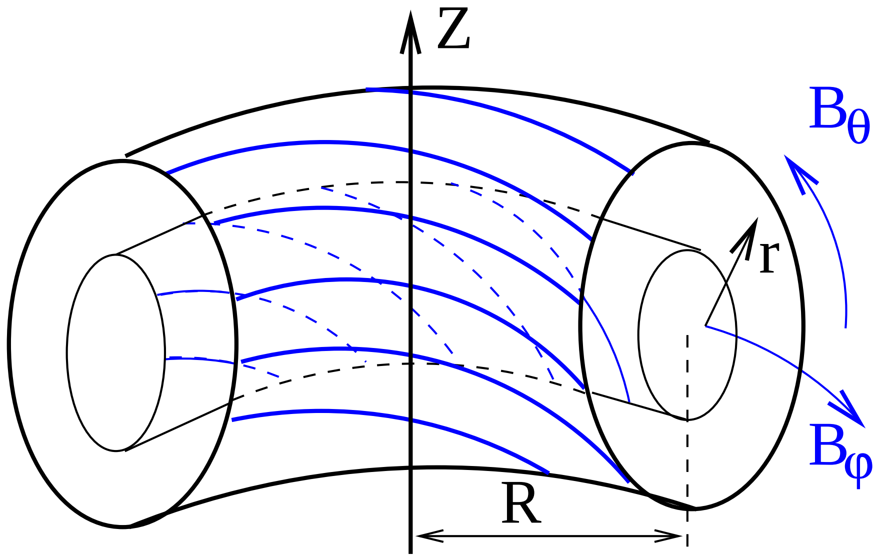
$$\gamma^2 = -v_A^2 k_\parallel^2 + v_A^2 \kappa \beta' \frac{k_y^2}{k_\perp^2} \quad \text{with} \quad \beta' = \frac{2\mu_0 p_0'}{B_0^2}$$

- instability if stabilization by field line bending ($v_A^2 k_\parallel^2$) is small:

$$\boxed{\kappa \beta' > k_\parallel^2} \qquad \text{more generally:} \quad \boxed{\boldsymbol{\kappa} \cdot \nabla \beta > k_\parallel^2}$$

Interchange mode in toroidal geometry

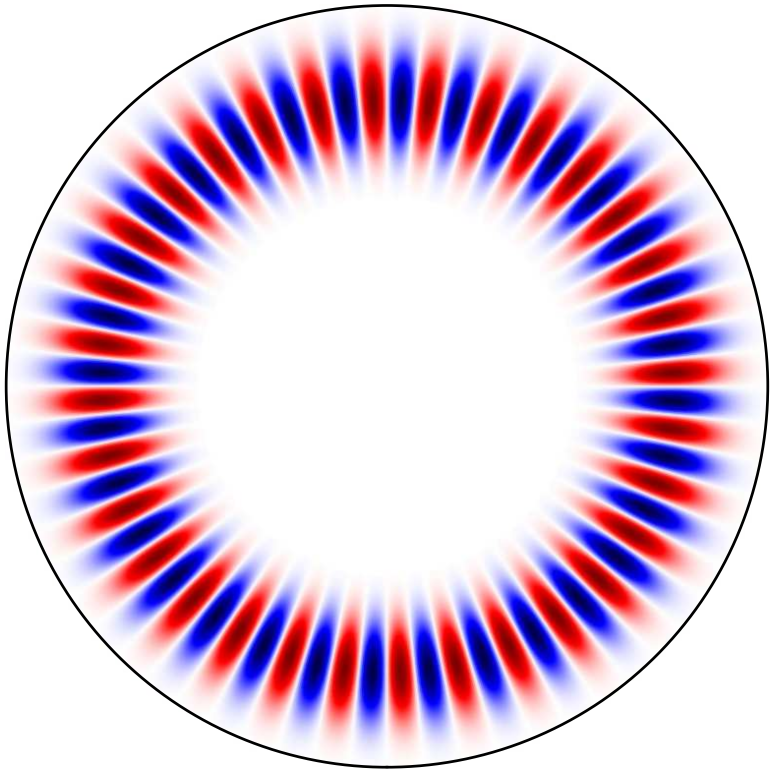
- equilibrium pressure gradient: $\nabla\beta \sim -\mathbf{e}_r$
- main contribution to magnetic curvature comes from toroidal field: $\boldsymbol{\kappa} = -\frac{1}{R}\mathbf{e}_R$
- interchange modes $\sim e^{im\theta}$ "feel" average curvature: $\langle \boldsymbol{\kappa}\cdot\mathbf{e}_r\rangle_{\theta\varphi} = \frac{-r}{(qR_0)^2}\left(1-q^2\right)$

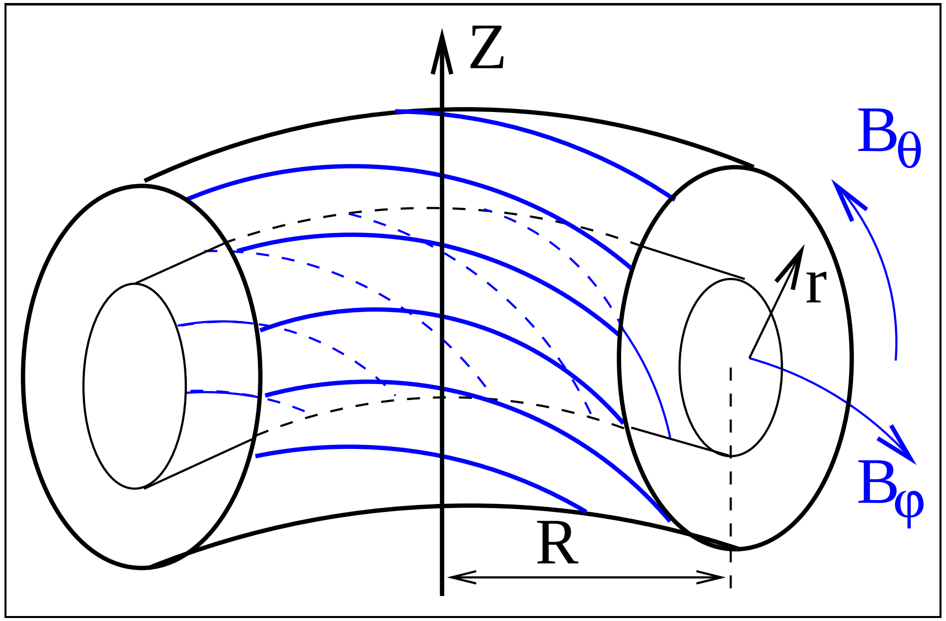


- estimation for $k_\parallel$: $\mathrm{i}k_\parallel\tilde{\phi} = \nabla_\parallel\tilde{\phi} \approx \frac{\mathrm{i}}{R_0}\left(\frac{m}{q}-n\right)\tilde{\phi}$, $\quad \frac{1}{q} \approx \frac{1}{q(r_s)} - \frac{q'}{q^2}\bigg|_{r_s}(r-r_s)$

$$\rightarrow \quad k_\parallel\tilde{\phi} \approx -\frac{m}{r_s}(r-r_s)\frac{s}{R_0 q}\tilde{\phi} \quad \rightarrow \quad \boxed{k_\parallel \sim -\frac{s}{R_0 q}} \quad \text{with} \quad s = \frac{r}{q}\frac{\mathrm{d}q}{\mathrm{d}r}$$

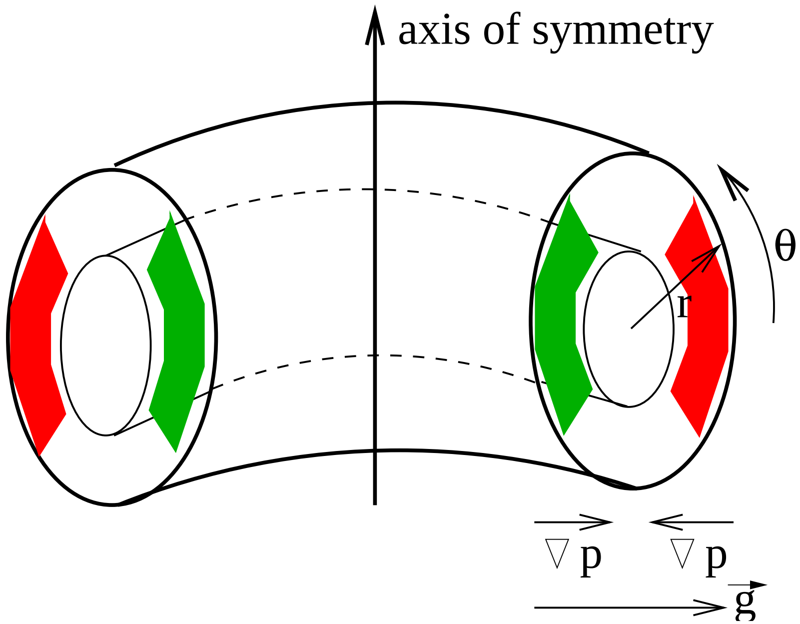
Interchange modes are stable if $\mathbf{q > 1}$





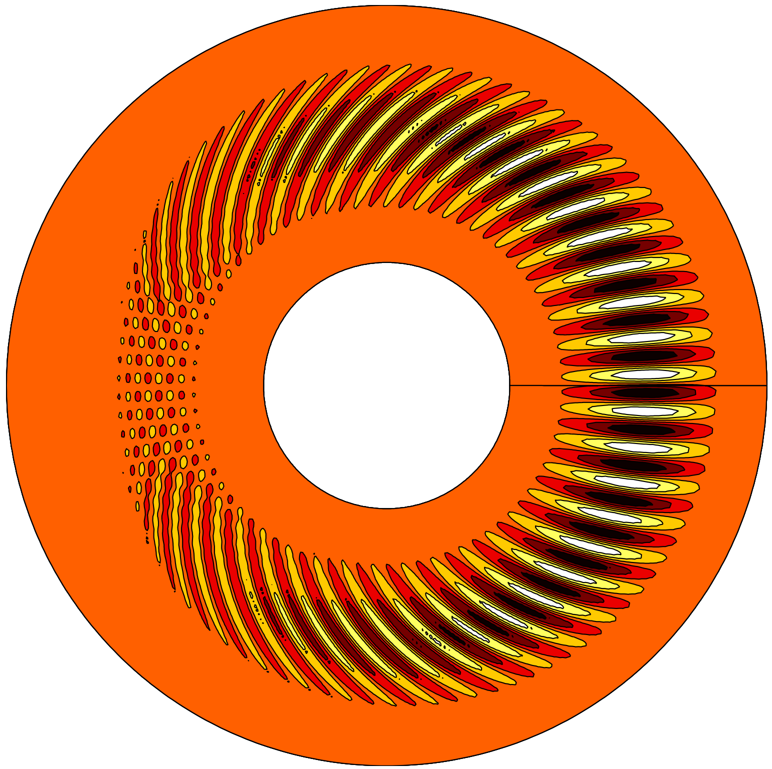
stable if $\boxed{\langle \boldsymbol{\kappa} \cdot \mathbf{e}_r \rangle_{\theta\varphi} \beta' < k_\parallel^2}$ $\rightarrow$ $\boxed{\frac{r}{q^2 R_0^2}(1 - q^2)\beta' < \frac{1}{4}\frac{s^2}{q^2 R_0^2}}$ (Mercier criterion)

Ballooning modes can be unstable even if $\mathbf{q > 1}$



in poloidal plane (r, θ),

magn. curvature depends on θ



electrostatic potential ϕ

of a $n = 18$ ballooning mode

(radial dim. stretched $\times$ 4)

Stability limit of ballooning mode

expressed as function of

- magnetic shear

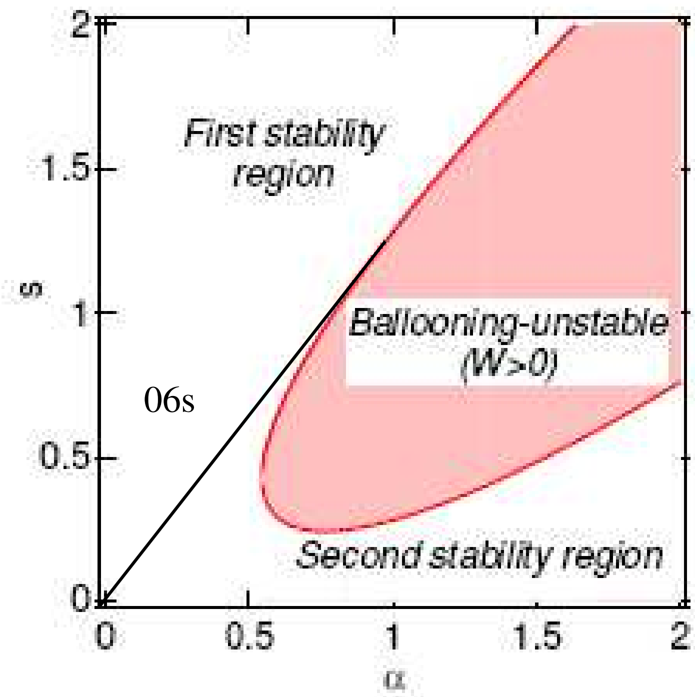
$$s = \frac{r}{q}\frac{\mathrm{d}q}{\mathrm{d}r}$$

- normalized pressure gradient

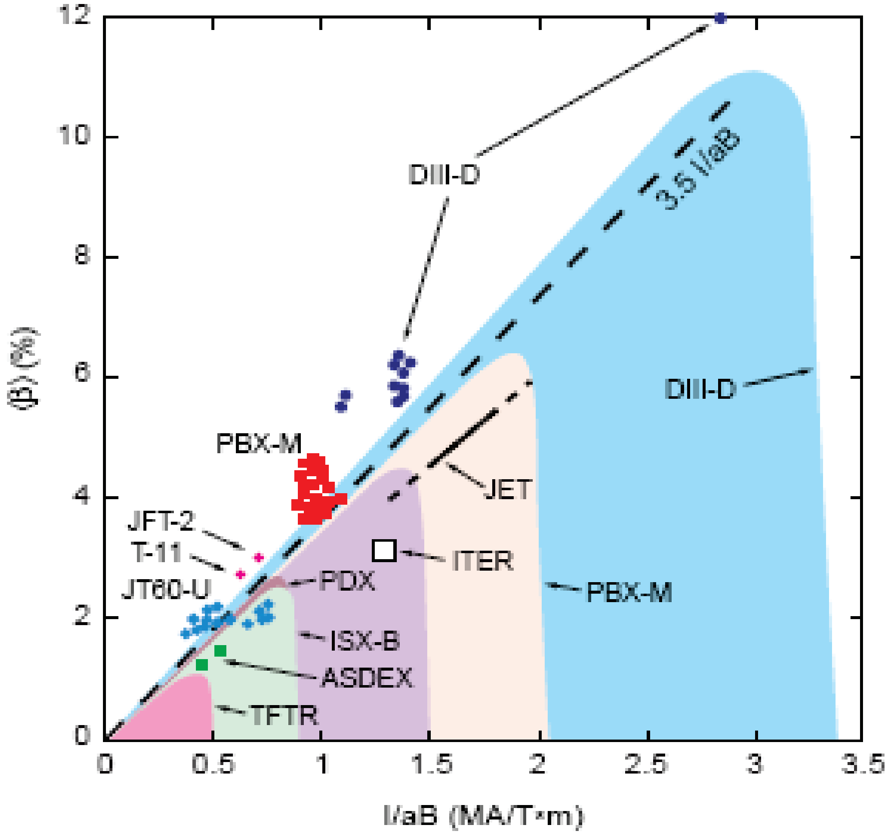
$$\alpha = -Rq^2\frac{\mathrm{d}\beta}{\mathrm{d}r}$$

- first stability region

$$\boxed{\alpha < 0.6s}$$



Ideal MHD beta limit

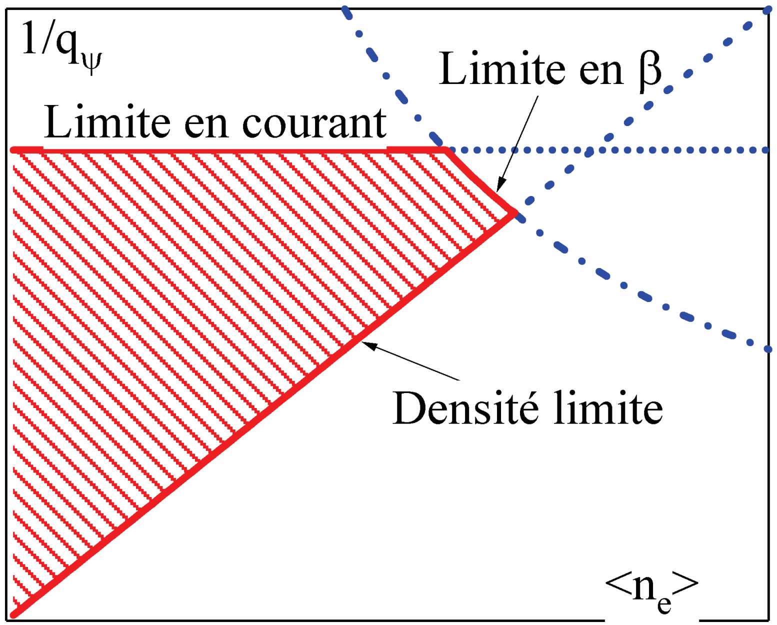


- for ballooning modes, $\alpha < 0.6s$ leads to:

$$\beta_N = \frac{\beta[\%]a[\mathrm{m}]B[\mathrm{T}]}{I[\mathrm{MA}]} < 3.5$$

- rather complete stability analysis including kink modes $\rightarrow$ $\beta_N < 2.8$ (Troyon limit)

Operational domain for tokamaks



[density limit (Greenwald): $\boxed{\bar{n}_e\left[10^{20}\,\mathrm{m}^{-3}\right] < \frac{I\,[\mathrm{MA}]}{\pi a^2\,[\mathrm{m}^2]}}$]

Lecture on **IDEAL MHD STABILITY**

Additional notes

P. Beyer / Aix Marseille University

Basic view of a kink instability

(refering to p. 7 of the lecture)

The current flowing in the twisted wire is

$$\mathbf{I} = I_0 \frac{\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}z}}{\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}z}\right|} = \frac{I_0}{\sqrt{1+\xi^2 k^2}} \left[-\xi k \sin(kz)\mathbf{e}_x + \xi k \cos(kz)\mathbf{e}_y + \mathbf{e}_z\right] \ .$$

The Lorentz force acting on the twisted wire is

$$\mathbf{I} \times \mathbf{B} = \frac{I_0 B}{\sqrt{1+\xi^2 k^2}} \left[\xi k \cos(kz)\mathbf{e}_x + \xi k \sin(kz)\mathbf{e}_y\right]$$

The equation of motion of the twisted wire is

$$\mu \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \mu \left[\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\cos(kz)\mathbf{e}_x + \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\sin(kz)\mathbf{e}_y\right] = \mathbf{I} \times \mathbf{B} \ .$$

where μ is the mass per unit length of the wire. The equation of motion for the deformation amplitude therefore is

$$\mu \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = \frac{I_0 B k}{\sqrt{1+\xi^2 k^2}} \xi \ .$$

For small deformations such that $\xi^2 k^2 \ll 1$, the amplitude grows exponentially $\xi \sim e^{\gamma t}$ with

$$\gamma^2 = \frac{I_0 B k}{\mu} \ .$$

Basic view of an interchange instability

(refering to p. 15, 16 of the lecture)

With

- $\left(\frac{\mathbf{B}}{B^2} \times \boldsymbol{\kappa}\right) \cdot \nabla p = \frac{\kappa}{B}\left(\mathbf{e}_z \times \mathbf{e}_x\right) \cdot \nabla p = \frac{\kappa}{B}\frac{\partial p}{\partial y} = \mathrm{i}\frac{\kappa}{B} k_y \tilde{p}$
- $\mathbf{u}_E \cdot \nabla p = \left(\frac{\mathbf{B}}{B^2} \times \nabla \phi\right) \cdot \nabla p \approx \left(\frac{\mathbf{B}}{B^2} \times \nabla \tilde{\phi}\right) \cdot p_0' \mathbf{e}_x = -\mathrm{i}\frac{p_0'}{B} k_y \tilde{\phi}$

the charge balance, Ohm's Law and pressure balance give

$$-\frac{n m_i}{B^2}\gamma k_\perp^2 \tilde{\phi} = -\mathrm{i} k_\parallel \frac{k_\perp^2}{\mu_0}\tilde{\psi} + 2\mathrm{i}\frac{\kappa}{B} k_y \tilde{p} \qquad (1)$$

$$\gamma \tilde{\psi} = \mathrm{i} k_\parallel \tilde{\phi} \qquad (2)$$

$$\gamma \tilde{p} = \mathrm{i}\frac{p_0'}{B} k_y \tilde{\phi} + \mathrm{i}\frac{10}{3} p_0 \frac{\kappa}{B} k_y \tilde{\phi} = \mathrm{i}\frac{p_0' + \frac{10}{3} p_0 \kappa}{B} k_y \tilde{\phi} \qquad (3)$$

where $k_\perp^2 = k_x^2 + k_y^2$. Using (2) and (3) to replace $\tilde{\psi}$ and $\tilde{p}$ in (1), respectively, one obtains the dispersion relation

$$\left[-\frac{nm_i}{B^2}\gamma k_\perp^2 + \mathrm{i}k_\parallel \frac{k_\perp^2}{\mu_0}\frac{\mathrm{i}k_\parallel}{\gamma} - 2\mathrm{i}\frac{\kappa}{B}k_y \frac{\mathrm{i}\left(p_0' + \frac{10}{3}p_0\kappa\right)}{\gamma B}k_y\right]\tilde{\phi} = 0$$
$$\rightarrow \quad -\frac{nm_i k_\perp^2}{B^2}\gamma^2 = k_\parallel^2\frac{k_\perp^2}{\mu_0} - 2\frac{\kappa\left(p_0' + \frac{10}{3}p_0\kappa\right)}{B^2}k_y^2$$
$$\rightarrow \quad \gamma^2 = -\frac{B^2}{\mu_0 n m_i}k_\parallel^2 + \frac{2\kappa\left(p_0' + \frac{10}{3}p_0\kappa\right)}{nm_i}\frac{k_y^2}{k_\perp^2} = -v_A^2 k_\parallel^2 + v_A^2\kappa\beta'\frac{k_y^2}{k_\perp^2}$$

where in the last step the Alfvén velocity $v_A = B/\sqrt{\mu_0 n m_i}$ and the plasma beta gradient $\beta' = 2\mu_0 p_0'/B^2$ have been introduced. Addtionally, $p_0\kappa \ll p_0'$ has been used as $p_0\kappa \sim p_0/R$ and $p_0' \sim p_0/L_p$ with $L_p \ll R$.