Lagrangian and geometrical methods in the fundamental physics of waves and their application to plasma dynamics

I. Y. Dodin

Princeton Plasma Physics Laboratory

- Fundamental wave equations. Geometrical optics from a Lagrangian
- Examples: (i) *linear* plasma waves, (ii) *nonlinear* waves with trapped particles
- Beyond geometrical optics: wave kinetics as quantum theory

6th ITER International School: RF heating and current drive in plasmas

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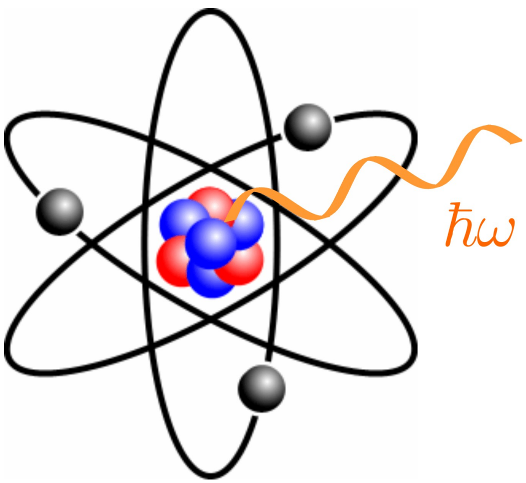
Simple question: what is the wave momentum in a medium?

- Surprisingly, the *answer* is not so simple... Consider absorption, $\Omega \approx \omega - kv$



Barnett, 2010; *Bradshaw et al., 2010*...

cf., e.g., *Cary and Kaufman, 1980*

$$N_{\rm ph} = \mathcal{E}/\hbar\omega$$

- Energy conservation:

$$^1\!/_2 M v^2 + \hbar\omega = {}^1\!/_2 M v'^2 + \hbar\Omega$$

$$\hookrightarrow\ ^1\!/_2 M(v'^2 - v^2) = \hbar(\omega - \Omega) = \hbar k v$$

- Momentum conservation:

$$M v + p_{\rm ph} = M v'$$

$$\hookrightarrow\ ^1\!/_2 M(v'^2 - v^2) \approx M(v' - v)v = p_{\rm ph} v$$

$$p_{\rm ph} = \hbar k, \qquad \mathcal{P} = k\mathcal{E}/\omega$$

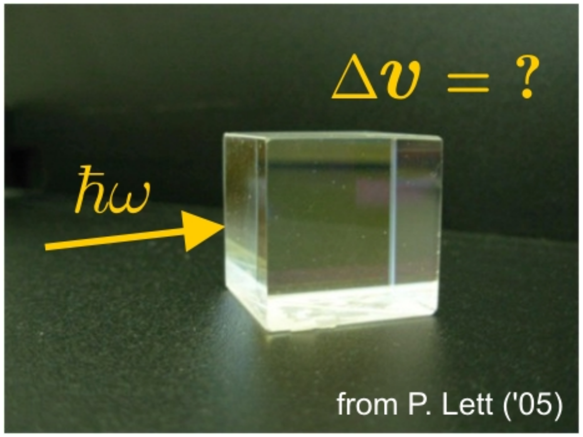
- But take a Langmuir wave... Why not $\mathcal{P} = 0$ when plasma is at rest?

Any alternative arguments? Well...

- Momentum conservation requires

$$\hbar\omega/c = p_{\rm ph} + MV$$

$$p_{\rm ph} = \hbar\omega/c - M\,\Delta x/\tau$$



- The center of mass shifts as without dielectric

$$m_{\rm ph} v_{\rm g}\tau + M\,\Delta x = m_{\rm ph} c\tau$$

$$\hookrightarrow \quad \Delta x = (c - v_{\rm g})\tau m_{\rm ph}/M$$

$$\omega = kV_{\rm A}, \quad \mathbf{k} \,||\, \mathbf{B}_0$$

$$v_{\rm ph} = v_{\rm g} = V_{\rm A}$$

Dewar, 1970

- Taking $m_{\rm ph} = \hbar\omega/c^2$ (in vacuum), we thus get

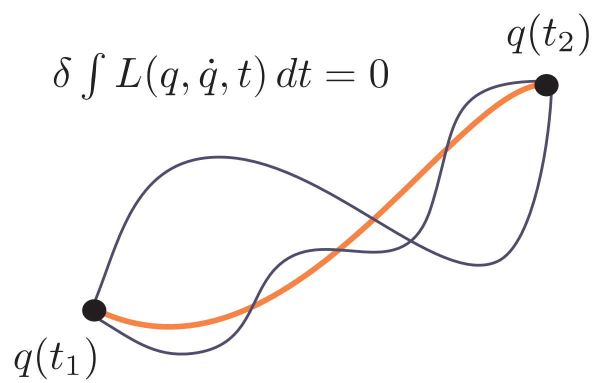
$$p_{\rm ph} = \hbar\omega v_{\rm g}/c^2, \qquad \mathcal{P} = v_{\rm g}\mathcal{E}/c^2$$

- A little problem, though... For Alfvén waves, one gets $\mathcal{P} = v_{\rm g}\mathcal{E}/(2c^2)$ instead!

$\partial_t \mathcal{P} + \partial_x(v_{\rm g}\mathcal{P}) =$ "force"... But there are *infinitely many* equations like this!

Least action principle for waves

- In classical physics, the least action principle is fundamental. Average action:



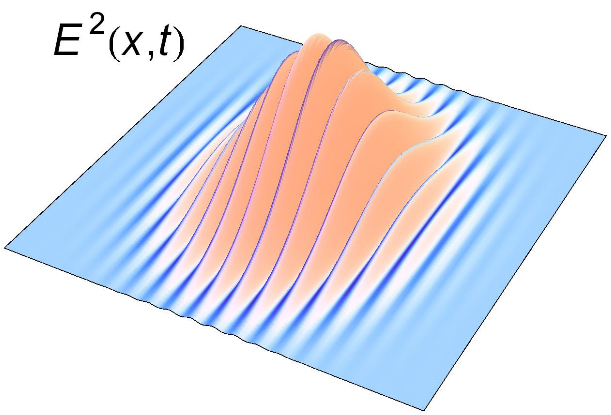
$$0 = \delta \int \mathcal{L}\, dt\, dx = \delta \int \left[\langle \mathcal{L} \rangle + \tilde{\mathcal{L}}\,\right] dt\, dx \approx \delta \int \langle \mathcal{L} \rangle\, dt\, dx$$

- **Definition** of the geometrical-optics limit:

  $\mathfrak{L}$ does not depend on gradients of (a, ω, k)

$$\langle \mathcal{L} \rangle = \mathfrak{L}(a, \omega, k;\, t, x)$$

$$\omega = -\partial_t \theta, \quad k = \partial_x \theta$$

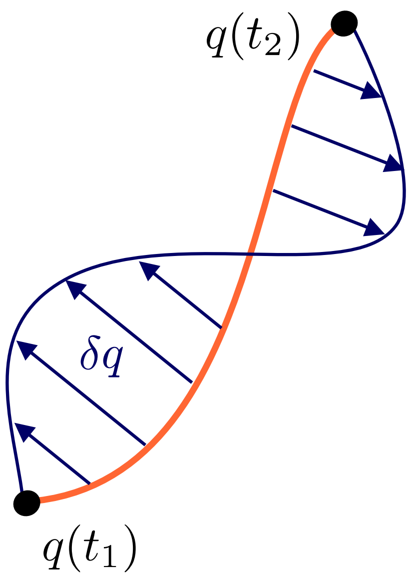


- Least action principle in geometrical optics:

$$\delta \int \mathfrak{L}(a, \omega, k)\, dt\, dx = 0$$

Sturrock, 1961; *Whitham, 1965*; *Whitham, 1974*; *Dodin and Fisch, 2012d* . . .

The three main (Whitham's) equations



- Proceed much like with a discrete system, $L = L(q, \dot{q})$

$$\delta L(q, \dot{q}) = L(q + \delta q, \dot{q} + \delta\dot{q}) - L(q, \dot{q}) = L_q\,\delta q + L_{\dot{q}}\,\delta\dot{q}$$

$$0 = \int_{t_1}^{t_2} (L_q\,\delta q + L_{\dot{q}} \underbrace{\delta\dot{q}}_{d_t\,\delta q})\,dt = L_{\dot{q}}\,\delta q\big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \underbrace{(L_q - d_t L_{\dot{q}})}_{\text{must be zero}} \underbrace{\delta q}_{\text{any}}\,dt$$

- First, we vary the wave action with respect to a:

$$0 = \delta_a \int \mathfrak{L}(a, \omega, k)\,dt\,dx = \int \underbrace{\mathfrak{L}_a}_{\text{must be zero}} \delta a\,dt\,dx$$

- Second, vary it with respect to θ, with $\omega = -\partial_t\theta$ and $k = \partial_x\theta$ in mind:

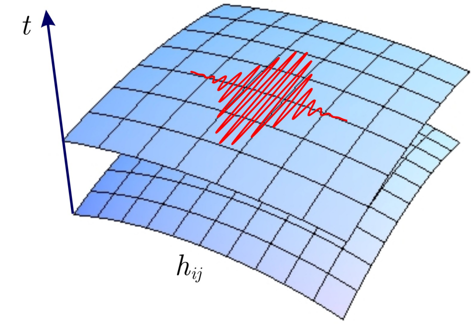
$$0 = \delta_\theta \int \mathfrak{L}(a, \omega, k)\,dt\,dx = \int \left[\mathfrak{L}_\omega\,\delta(-\theta_t) + \mathfrak{L}_k\,\delta\theta_x\right] dt\,dx = \int \underbrace{(\partial_t\mathfrak{L}_\omega - \partial_x\mathfrak{L}_k)}_{\text{must be zero}} \delta\theta\,dt\,dx$$

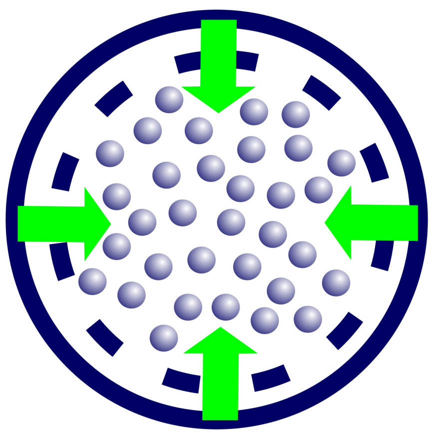
- Third, $\partial_t k = -\partial_x\omega$, as seen from the definitions of ω and k and $\partial^2_{xt}\theta = \partial^2_{tx}\theta$

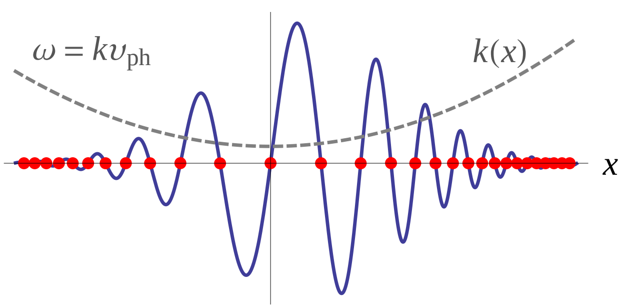
But what do Whitham's equations *mean*?

$$\mathfrak{L}_a(a,\omega,k)=0,\quad \partial_t\mathfrak{L}_\omega-\partial_x\mathfrak{L}_k=0,\quad \partial_t k+\partial_x\omega=0$$

$$\mathfrak{L}_a(a,k_\mu)=0,\qquad (\mathfrak{L}_{k_\mu})_{;\mu}=0,\qquad k_{\mu;\nu}=k_{\nu;\mu}$$







1. Nonlinear dispersion relation
2. Action conservation, $\int \mathfrak{L}_\omega\, dx = \text{const}$ (same as photon conservation in QM)

   cf. $\partial_t n + \partial_x(nv) = 0$

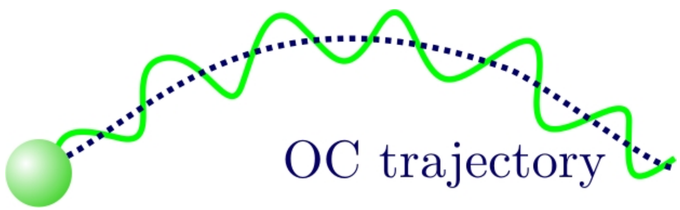
   - At $\partial_x = 0$, one has $\mathfrak{L}_\omega = \text{const}$, whence $a = a(t)$
   - At $\partial_t = 0$, one has $\mathfrak{L}_k = \text{const}$, whence $a = a(x)$
3. Consistency equation ∼ "crest conservation"

Whitham's equations are complete and can be used in numerical simulations

Finding the Lagrangian for a wave in plasma

$$L_{\text{total}} = \int \frac{E^2 - B^2}{8\pi}\,dx + \sum_i L_i(x_i, v_i) \xrightarrow{\text{averaging}} \int \frac{\langle E^2 - B^2 \rangle}{8\pi}\,dx + \sum_i \underbrace{\langle L_i(X + \tilde{x}, V + \tilde{v}) \rangle}_{\langle L \rangle_i (X_i, V_i)}$$

- $\langle L \rangle (X, V)$ is the slow-motion, or oscillation-center Lagrangian of a single particle



Dodin and Fisch, 2012a; *Dodin and Fisch, 2011*

- Use canonical variables, $(X, V) \to (X, P)$:

$$P \equiv \partial_V \langle L \rangle, \quad \mathcal{H}(X, P) = PV - \langle L \rangle$$

- Drop PV as independent of wave variables

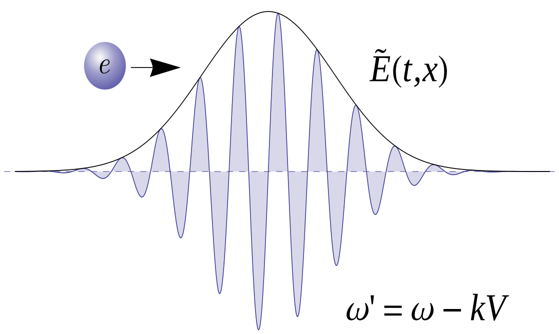
$$\langle L_{\text{total}} \rangle \longrightarrow L_{\text{wave}} = \int \frac{\langle E^2 - B^2 \rangle}{8\pi}\,dx + \underbrace{\sum\nolimits_i P_i V_i}_{\text{unimportant}} - \underbrace{\sum\nolimits_i \mathcal{H}_i(P_i, X_i)}_{N\langle \mathcal{H} \rangle}$$

$$\mathfrak{L}(a, \omega, \mathbf{k}) = \frac{\langle E^2 - B^2 \rangle}{8\pi} - \sum_s n_s \langle \mathcal{H}_s \rangle_P$$

- Need to find only single-particle $\mathcal{H}$
- Fully nonlinear description
- Vlasov equation not needed

Example: $\mathcal{H}$ in linear waves. Ponderomotive potential Φ

- Consider an electron in a 1D electrostatic wave, $L(x,v) = {}^1\!/_2\, mv^2 - e\tilde{\varphi}(t,x)$



$$x = X + \tilde{x}, \qquad \tilde{x} \approx -e\tilde{E}(t,X)/m\omega'^2$$

- Ponderomotive Lagrangian:

$$L = mV^2/2 + mV\tilde{v} + m\tilde{v}^2/2 - e\tilde{\varphi}(t,X) - e\,\overbrace{\partial_x\tilde{\varphi}(t,X)}^{-\tilde{E}}\,\tilde{x}$$

$$\langle L\rangle = mV^2/2 + \underbrace{m\langle\tilde{v}^2\rangle/2 + e\langle\tilde{E}\tilde{x}\rangle}_{-\Phi(t,X,V)}$$

$$\dot{P} = -\Phi_X, \quad (m - \Phi_{VV})\dot{V} = -\Phi_X + \Phi_{Vt} + V\Phi_{VX}$$

$$\Phi = \frac{e|\tilde{E}(t,X)|^2}{4m(\omega - kV)^2}$$

$$P = \partial_V\langle L\rangle$$

$$P = mV - \partial_V\Phi$$

- Ponderomotive Hamiltonian, $\mathcal{H} = PV - \langle L\rangle$:

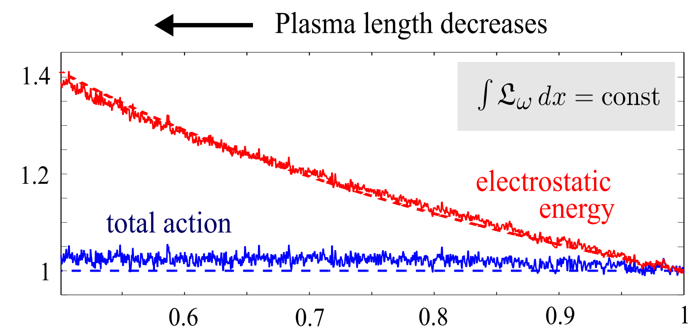
$$\mathcal{H} = mV^2/2 - V\Phi_V + \Phi = \underbrace{(mV - \Phi_V)^2/2m}_{P^2/2m} - \underbrace{\Phi_V^2/2m}_{\sim E^4} + \Phi$$

$$\mathcal{H} \approx P^2/2m + \Phi$$

...which leads to linear Langmuir waves!

$$\mathfrak{L} = \underbrace{\frac{\langle E^2\rangle}{8\pi}}_{|\tilde{E}|^2/16\pi} - \underbrace{\frac{\langle B^2\rangle}{8\pi}}_{0} - n\left\langle \underbrace{\frac{P^2}{2m}}_{\text{does not matter}} + \frac{e|\tilde{E}|^2}{4m(\omega - kV)^2}\right\rangle \rightarrow \frac{|\tilde{E}|^2}{16\pi} - n\left\langle \frac{e|\tilde{E}|^2}{4m(\omega - kV)^2}\right\rangle$$

$$\mathfrak{L} = \frac{|\tilde{E}|^2}{16\pi}\Bigg[\underbrace{1 - \frac{4\pi n e^2}{m}\int_{-\infty}^{+\infty}\frac{f_0(V)}{(\omega - kV)^2}\,dV}_{\text{this is known as the linear } \epsilon(\omega,k)!}\Bigg]$$



- $\mathfrak{L}_a = 0$ correctly predicts $\epsilon(\omega, k) = 0$

$$\mathfrak{L}_\omega \approx |\tilde{E}|^2/(8\pi\omega), \quad \mathfrak{L}_\omega \sim L^{-1} \sim n$$

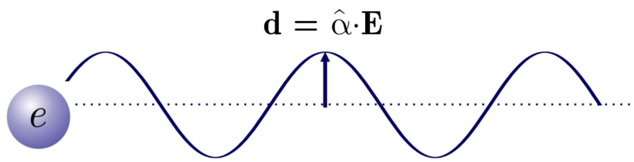
$$|\tilde{E}|^2 \sim n\omega \sim n^{3/2}$$

- $\partial_t \mathfrak{L}_\omega - \nabla \cdot \mathfrak{L}_{\mathbf{k}} = 0$ is *much* easier than

$$\frac{\partial'^2 \tilde{n}}{\partial t^2} + \omega_p^2 \tilde{n} - C_{j\ell}\frac{\partial^2 \tilde{n}}{\partial x_j\,\partial x_\ell} + 2\frac{\partial' \tilde{n}}{\partial t}\left(\frac{\Omega}{\omega_p}\frac{\partial \omega_p}{\partial x_\ell} + k_j \frac{\partial u_j}{\partial x_\ell}\right)\frac{k_\ell}{k^2} - \left(\delta_{js} + \frac{k_j k_s}{k^2}\right)\frac{\partial C_{s\ell}}{\partial x_j}\frac{\partial \tilde{n}}{\partial x_\ell} = 0$$

Schmit et al., 2010; *Dodin et al., 2009*

Now let's make it a bit more general... Any linear waves



$$\mathfrak{L} = \frac{1}{16\pi}\left(\tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \cdot \tilde{\mathbf{E}} - \frac{c^2}{\omega^2}|\mathbf{k} \times \tilde{\mathbf{E}}|^2\right)$$

- The so-called K-χ theorem:

  Cary and Kaufman, 1977, 1981; Dodin and Fisch, 2010b

$$\mathcal{H} \approx P^2/2m + \Phi, \quad \Phi = -{}^1\!/_4\, \mathbf{E}^* \cdot \hat{\alpha} \cdot \mathbf{E}$$

(Φ is the dipole interaction energy)

$$\mathfrak{L} = \underbrace{\frac{\langle \tilde{E}^2 \rangle}{8\pi}}_{|E|^2/16\pi} - \underbrace{\frac{\langle \tilde{B}^2 \rangle}{8\pi}}_{|B|^2/16\pi} + \sum_s \frac{n_s}{4}\mathbf{E}^* \cdot \langle \hat{\alpha}_s \rangle \cdot \mathbf{E} = \frac{|\tilde{E}|^2}{16\pi} - \underbrace{\frac{|\tilde{B}|^2}{16\pi}}_{\tilde{B} = c\mathbf{k}\times\tilde{\mathbf{E}}/\omega} + \mathbf{E}^* \cdot \underbrace{\left(\frac{1}{4}\sum_s n_s \langle \hat{\alpha}_s \rangle\right)}_{=(\hat{\epsilon}-1)/16\pi} \cdot \mathbf{E}$$

- All linear waves have $\mathfrak{L} = a^2\mathfrak{D}(\omega, \mathbf{k})$, for given polarization

$$\mathfrak{L}_\omega = a^2\mathfrak{D}_\omega, \quad \mathfrak{L}_\mathbf{k} = a^2\mathfrak{D}_\mathbf{k}$$

$$-\mathfrak{L}_\mathbf{k} = -(\mathfrak{D}_\mathbf{k}/\mathfrak{D}_\omega)\mathfrak{L}_\omega = \omega_\mathbf{k}\mathfrak{L}_\omega$$

$$\mathfrak{D}(\omega, \mathbf{k}) = 0, \quad \mathfrak{D}_\mathbf{k} + \mathfrak{D}_\omega\, \omega_\mathbf{k} = 0$$

$$\partial_t \mathfrak{L}_\omega + \nabla \cdot (\mathbf{v}_\mathrm{g}\mathfrak{L}_\omega) = 0$$

$$\partial_t(\mathsf{X}\mathfrak{L}_\omega) + \nabla \cdot (\mathbf{v}_\mathrm{g}\mathsf{X}\mathfrak{L}_\omega) = \ldots$$

So the energy-momentum tensor is...

$$\partial_t(\underbrace{\omega\mathfrak{L}_\omega}_{\text{energy}}) + \nabla\cdot(\underbrace{\mathbf{v}_\mathrm{g}\omega\mathfrak{L}_\omega}_{\text{energy flux}}) = -\mathfrak{L}_t$$

$$\partial_t(\underbrace{\mathbf{k}\mathfrak{L}_\omega}_{\text{momentum}}) + \nabla\cdot(\underbrace{\mathbf{v}_\mathrm{g}\mathbf{k}\mathfrak{L}_\omega}_{\text{momentum flux}}) = \mathfrak{L}_\mathbf{x}$$

Sturrock, 1961; *Whitham, 1965*

$$T^{\alpha\beta}_{\text{can}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}\mathbf{v}_g/c \\ \mathcal{E}\mathbf{k}c/\omega & \mathcal{E}\mathbf{k}\mathbf{v}_g/\omega \end{pmatrix}$$

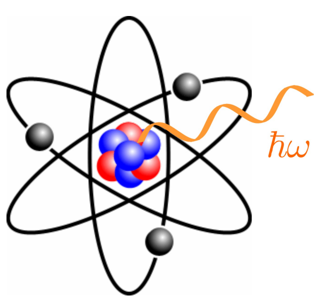
any linear wave; any frame

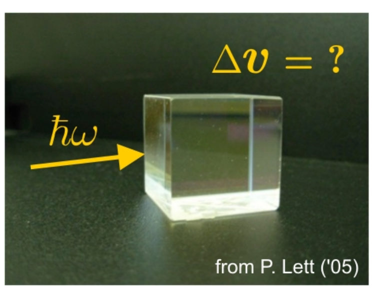
- So, after all, $\boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{E}/\omega$? Well...
  - Each particle carries $\mathbf{P} \approx m\mathbf{V} - \partial_\mathbf{V}\Phi$
  - So, alternatively, $\bar{\boldsymbol{\mathcal{P}}} \equiv \boldsymbol{\mathcal{P}} - n\,\partial_\mathbf{V}\Phi$ belongs to the wave, and $m\mathbf{V}$ belong to particles

Dodin and Fisch, 2012d; *Dewar, 1977*

$$T'^{\alpha\beta}_{\text{full}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}\mathbf{v}_g/c \\ \mathcal{E}\mathbf{v}_g/c & \mathcal{E}\,\mathbf{k}\mathbf{v}_g/\omega + n\Phi \end{pmatrix}'$$

any linear wave *in isotropic fluid*; rest frame



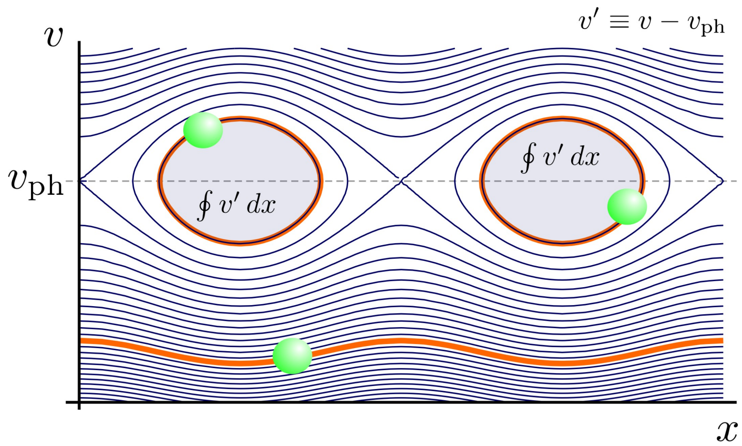


- This new momentum happens to be

$$\bar{\boldsymbol{\mathcal{P}}} \approx \mathbf{v}_\mathrm{g}\mathcal{E}/c^2$$

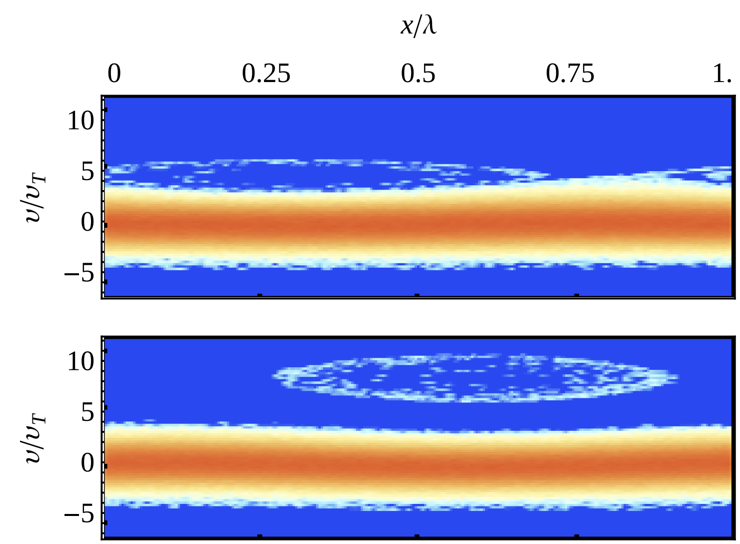
- Interestingly, electromagnetism is irrelevant
  - same for angular momenta, photon spin. . .

BGK modes: very special nonlinearity



- Bernstein-Greene-Kruskal waves
  Bernstein et al., 1957 . . .
  - Electrons with $v \approx \omega/k$ are trapped
  - They follow the island wherever it goes
- Trapped particles conserve $\oint (v - v_{\mathrm{ph}})\,dx$

- E.g., in plasma compressing $\perp \mathbf{k}$
  - $\omega \sim \omega_p \propto \sqrt{n(t)}$, $k =$ const $\rightarrow$
- Fundamental problems:
  - Conservation laws, stability
  - Nonlinear dispersion relations



Dodin and Fisch, 2012a; *Dodin and Fisch, 2012b*; *Dodin and Fisch, 2012c*; *Dodin and Fisch, 2011*

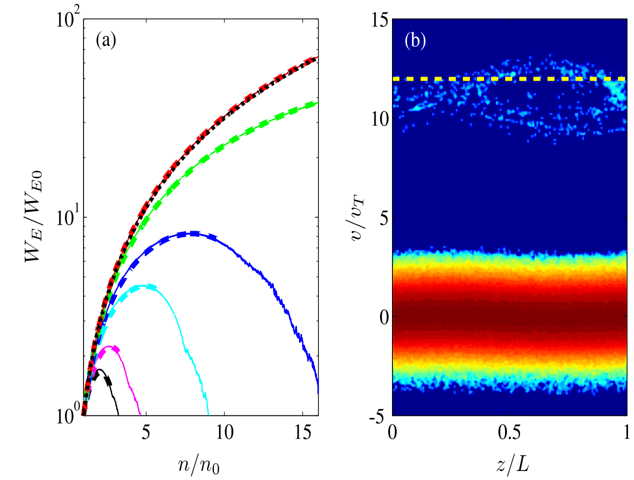
BGK wave amplification

- **Approximation 1:** the *bulk*-plasma response is linear

$$\mathfrak{L} \approx \epsilon(\omega, k)\,\frac{E_\mathrm{m}^2}{16\pi} - n_\mathrm{t}\,\langle \varepsilon_\mathrm{t}(k, E_\mathrm{m})\rangle + \frac{m n_\mathrm{t}}{2}\left(\frac{\omega}{k}\right)^2$$

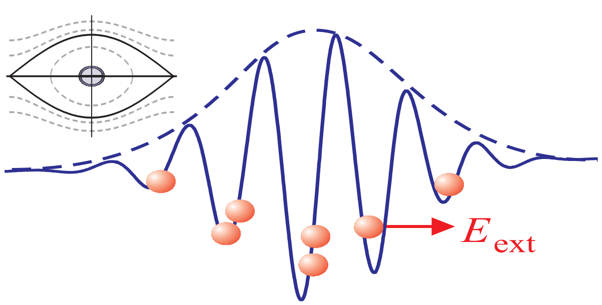
↳ Action conservation gives $E_\mathrm{m}(t)$ (cf. PIC results!):

$$\text{const} = \int \mathfrak{L}_\omega\, d^3x \sim \left[\epsilon_\omega E_\mathrm{m}^2/(16\pi) + m n_\mathrm{t} v_\mathrm{ph}^2\right]/n$$



- **Approximation 2:** particles are *deeply* trapped, so $\varepsilon_\mathrm{t} = -eE_\mathrm{m}/k - e\varphi_\mathrm{ext}$

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_\omega E_\mathrm{m}^2}{16\pi} + \sigma m v_\mathrm{ph}\right) + \frac{\partial}{\partial x}\left(-\frac{\epsilon_k E_\mathrm{m}^2}{16\pi} + \frac{1}{2}\sigma m v_\mathrm{ph}^2\right) = -\sigma e E_\mathrm{ext}$$



↑ $\sigma \equiv n_\mathrm{t}/k$ is the number of particles per wavelength

$$n_\mathrm{t} = k\sigma = k \times \text{const}$$

- The dc force performs work but cannot change $\langle v_\mathrm{t}\rangle = v_\mathrm{ph}$

↳ The energy must be spent on wave amplification!

Nonlinear group velocities. Modulational instability

$$\partial_t \underbrace{\mathfrak{L}_\omega}_{\mathcal{I}} + \partial_x \underbrace{(-\mathfrak{L}_k)}_{\mathcal{J}} = 0, \quad \partial_t k + \partial_x \omega = 0$$

- Let all depend on $\xi(t,x) \equiv x - Y(t)$

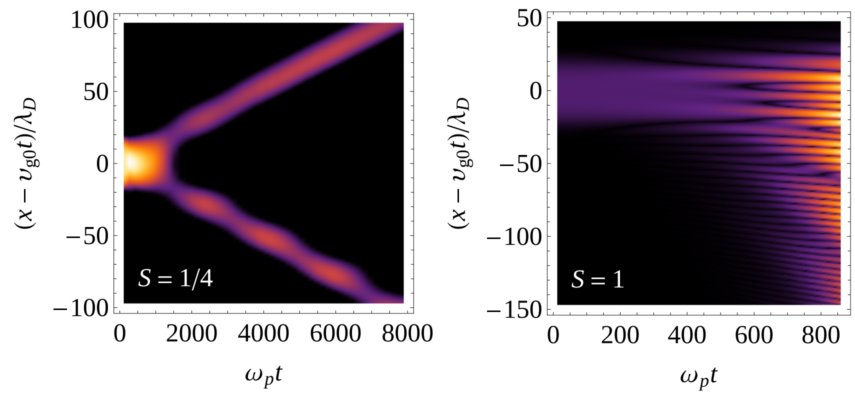
$$v_\mathrm{g} \equiv \dot{Y}, \quad \partial_x = d_\xi, \quad \partial_t = -v_\mathrm{g} d_\xi$$

$$v_\mathrm{g}(k, \mathcal{I}) = \omega_k \pm \sqrt{\omega_\mathcal{I} \mathcal{J}_k}$$

↳ $\omega_\mathcal{I} \mathcal{J}_k > 0$: stable wave, pulse splitting

↳ $\omega_\mathcal{I} \mathcal{J}_k < 0$: instability, $\gamma = |\Delta v_\mathrm{g}|\, \Delta k$

$$v_g \approx v_{g0}\left[1 \pm \Omega_E \sqrt{S\,(1/2 - S)}\,\right]$$



$$S = \frac{\text{trapped-}e \text{ energy flux}}{\text{passing-}e \text{ energy flux}}$$

- BGK waves do *not* satisfy the usual Schrödinger equation with *local* nonlinearity

$$\Delta\omega \approx \omega_k\, \Delta k + \tfrac{1}{2}\, \omega_{kk}\, (\Delta k)^2 \rightarrow i\partial_t, \quad \Delta k \rightarrow -i\partial_x$$

$$i(\partial_t + v_{\mathrm{g}0}\partial_x)\psi + \tfrac{1}{2}\, v'_{\mathrm{g}0}\, \partial^2_{xx}\psi = \omega_\mathrm{NL}(|\psi|^2)\psi$$

Dodin and Fisch, 2012c; *Lighthill, 1965*; *Whitham, 1974*; cf. *Dewar et al., 1972*; *Ikezi et al., 1978*; *Rose and Yin, 2008*...

BGK wave dispersion, $E \sim e^{ikx}$

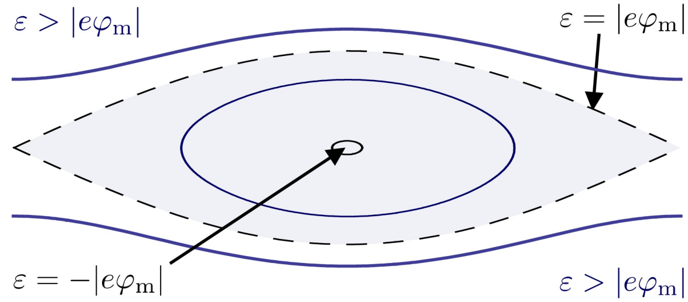
$$\mathfrak{L} = \langle \tilde{E}^2 \rangle / 8\pi \; - \; \left[n_\mathrm{p} \langle \varepsilon_\mathrm{p} + P v_\mathrm{ph} - m v_\mathrm{ph}^2/2 \rangle + n_\mathrm{t} \langle \varepsilon_\mathrm{t} \rangle \right] + n_\mathrm{t} m v_\mathrm{ph}^2/2$$

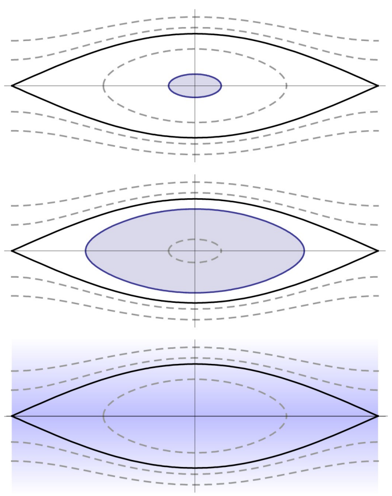
ε is the single-particle energy in the wave rest frame

$$0 = \mathfrak{L}_a = \partial_a \left[E_\mathrm{m}^2/16\pi - n\langle \varepsilon \rangle \right]$$

$$\boxed{\omega^2 = 2\omega_p^2 \, \partial_a \langle \varepsilon \rangle / (e\varphi_\mathrm{m})}$$

$a \equiv keE_\mathrm{m}/(m\omega^2)$



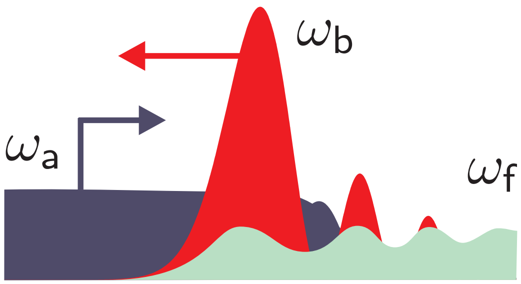


- Deeply trapped particles: $\omega^2 \approx \omega_\mathrm{L}^2 - 2\omega_\mathrm{t}^2 a^{-1}$
- Flat-top beam: $\omega^2 \approx \omega_\mathrm{L}^2 - \beta a^{-1/2}$
- Smooth distribution: $\omega_\mathrm{NL} \approx C_1 a^{1/2} + C_2 \ln a \ldots$

Dodin and Fisch, 2011; *Dodin and Fisch, 2012b*

cf. *Manheimer and Flynn, 1971*; *Dewar, 1972*; *Winjum et al., 2007*; *Khain and Friedland, 2007*; *Goldman and Berk, 1971*; *Krasovsky, 2007*; *Rose and Russell, 2001*; *Lindberg et al., 2007*...

Manley-Rowe relations



$$\omega_a \approx \omega_b + \omega_f$$

- Suppose two waves in resonance; then the *beat* phase, $\theta \equiv \theta_1 - \theta_2$, is a slow variable and may enter $\mathfrak{L}$:

$$\mathfrak{L} = \mathfrak{L}(\underbrace{a_1, \omega_1, k_1}_{\text{wave 1}}, \underbrace{a_2, \omega_2, k_2}_{\text{wave 2}}, \theta)$$

- Now vary $\mathfrak{L}$ with respect to θ_1 and θ_2 separately:

$$\delta_{\theta_1} \int \mathfrak{L}\, dt\, dx = \int \left[\mathfrak{L}_{\omega_1}\, \delta(-\theta_{1t}) + \mathfrak{L}_{k_1}\, \delta\theta_{1x} + \mathfrak{L}_\theta \delta\theta_1\right] dt\, dx = \int \underbrace{(\partial_t \mathfrak{L}_{\omega_1} - \partial_x \mathfrak{L}_{k_1} + \mathfrak{L}_\theta)}_{\text{this must be zero}}\, \delta\theta_1\, dt\, dx$$

$$\delta_{\theta_2} \int \mathfrak{L}\, dt\, dx = \int \left[\mathfrak{L}_{\omega_2}\, \delta(-\theta_{2t}) + \mathfrak{L}_{k_2}\, \delta\theta_{2x} - \mathfrak{L}_\theta \delta\theta_2\right] dt\, dx = \int \underbrace{(\partial_t \mathfrak{L}_{\omega_2} - \partial_x \mathfrak{L}_{k_2} - \mathfrak{L}_\theta)}_{\text{this must be zero}}\, \delta\theta_2\, dt\, dx$$

$$\partial_t(\mathfrak{L}_{\omega_1} + \mathfrak{L}_{\omega_2}) - \partial_x(\mathfrak{L}_{k_1} + \mathfrak{L}_{k_2}) = 0$$

$$\int (\mathfrak{L}_{\omega_1} + \mathfrak{L}_{\omega_2})\, dx = \text{const}$$

- Same for *three*-wave interactions ($\theta \equiv \theta_1 - \theta_2 - \theta_3$), etc.

Brizard and Kaufman, 1995;
Dodin et al., 2008; *Dodin and Fisch, 2008a*

General wave kinetics. Quantum-like geometry of waves

$$i\,|\dot{\psi}\rangle = \hat{\mathcal{H}}\,|\psi\rangle$$

$$i\dot{\hat{\rho}} + [\hat{\rho}, \hat{\mathcal{H}}] = 0$$

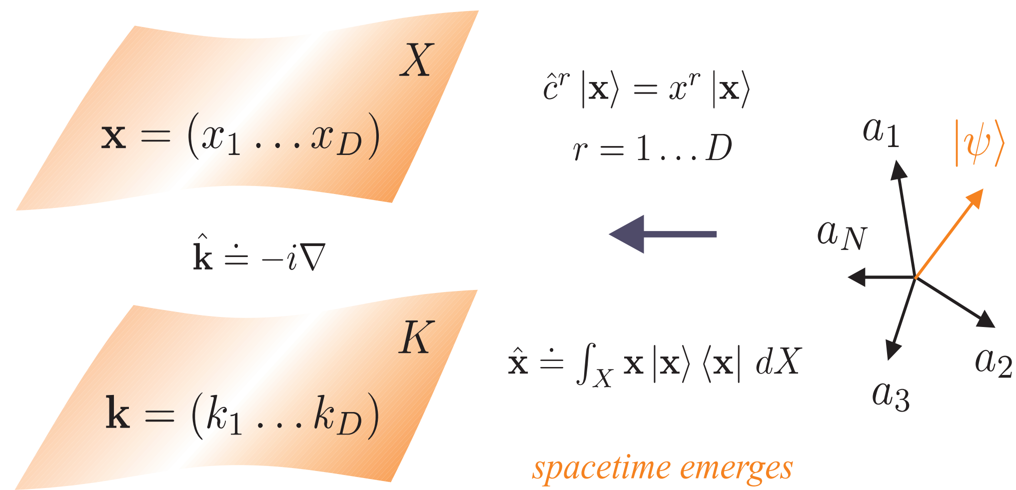
- Any oscillations, $L = \frac{1}{2}\,M_{ij}\dot{q}^i\dot{q}^j + R_{ij}\dot{q}^i q^j - \frac{1}{2}\,Q_{ij}q^i q^j$
  - ↳ $(q, p) \to (a, a^*)$, where $|a_n|^2$ are eigenmode actions
  - ↳ action is an operator, $\hat{\rho} \equiv |\psi\rangle\langle\psi|$, and $\operatorname{Tr}\hat{\rho} = \text{const}$

- Waves *never* decay (fields do)
  - radiation friction = Landau damping
  - hidden conservation laws

    cf. *Yakhot and Zakharov, 1993*

- If X is differentiable everywhere,

  then $\hat{\mathcal{H}} = \hat{\mathcal{H}}(\hat{\mathbf{x}}, \hat{\mathbf{k}})$ ↴



$$\partial_t f + \{\{f, \mathcal{H}\}\} = 0$$

cf. *McDonald and Kaufman, 1985*; *Brizard et al., 1993*

- f = averaged projector, not Winger function
- $(\mathbf{x}, \mathbf{k})$ are interpreted, not postulated

Summary

- Wave physics becomes more flexible/reliable within a Lagrangian approach
  - What is the wave momentum?.. energy?.. group velocity?.. transport equations?..
- For waves in plasma, $\mathfrak{L}$ is derived *from single-particle motion*:

$$\mathfrak{L}(a, \omega, \mathbf{k}) = \frac{\langle E^2 - B^2 \rangle}{8\pi} - \sum_s n_s \langle \mathcal{H}_s \rangle_f$$

- Showed applications to two types of waves:
  - textbook linear waves in plasma
  - BGK modes: dispersion, dynamics, stability...
- But those are just examples... The actual physics is about wave geometry!

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