# Collisional Transport in Tokamak Geometry

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# Outline of the lecture

- **Symmetries of a Tokamak plasma**
  - *The system of nested toroidal surfaces tangent to the magnetic field (magnetic surfaces)*
  - *Derivation of the most suitable coordinate system to describe functions in a space with toroidal symmetry: the toroidal coordinates*
  - *Introduction of the volume element and surface average in toroidal coordinates*
  - *Examples: calculation of the surface average of some functions*

- **The transport problem: time variation of particle and pressure in the volume enclosed by a toroidal (magnetic) surface**
  - *Calculation of classical fluxes*

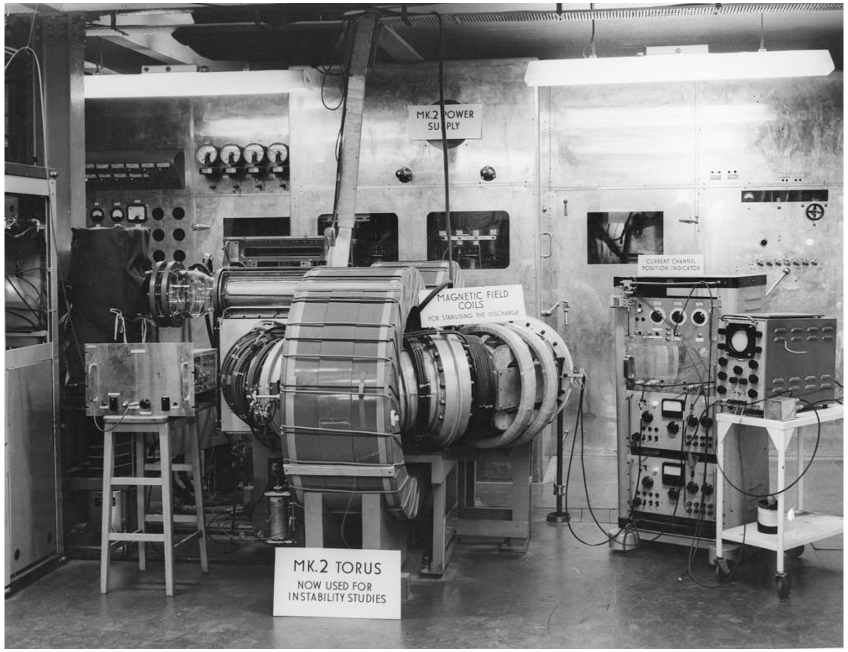
- **Derivation of the P-S current**
  - *Relation between radial pressure gradients and the P-S current*
  - *Relation between P-S current and the poloidal component of pressure gradient and electric field*
  - *Relation between the poloidal variation of the electron temperature and radial pressure gradients*

- **Calculation of the P-S fluxes**
  - *The classical diffusion coefficient and heat conductivities are enhanced by a factor $q^2$*

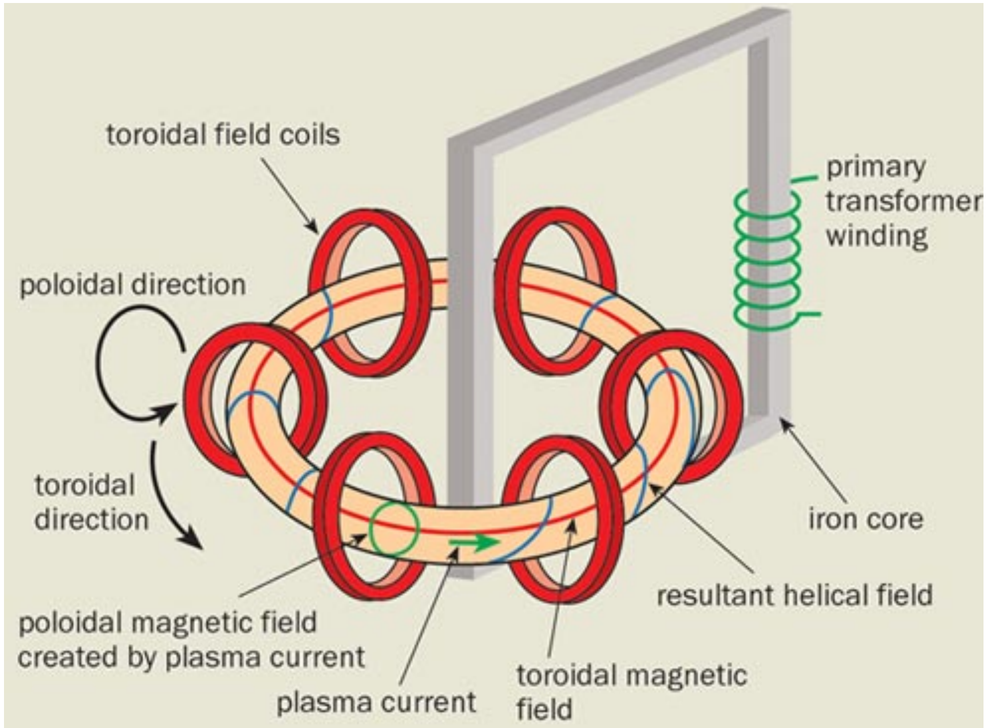
- **P-S Transport of heavy impurities in rotating plasmas**

References: *Rutherford, P-S, P.Helander, J Wesson, M Romanelli*

# Symmetries of a Tokamak plasma

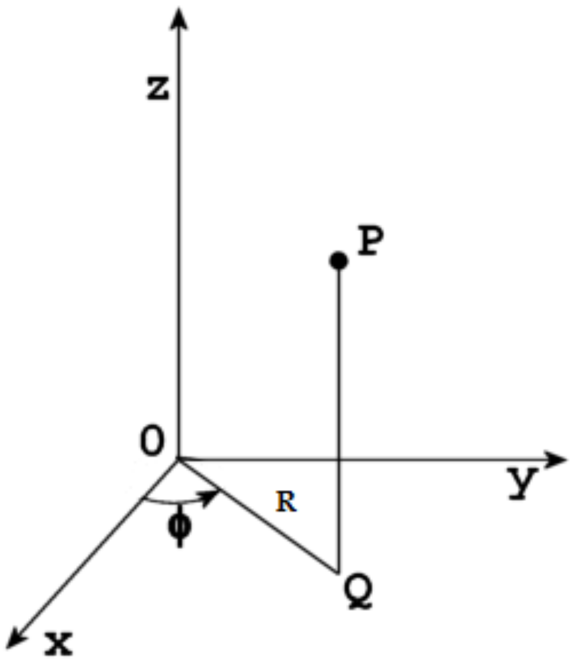



A tokamak, as we all know, is a toroidal vessel in which a strong toroidal magnetic field is produced by external coils:



# Symmetries of a Tokamak plasma

Let us consider a Cartesian coordinate system with the z axis coincident with the tokamak symmetry axis




- The magnetic field has cylindrical symmetry and can be conveniently written in cylindrical coordinates

$$B = B_x \vec{i} + B_y \vec{j} + B_z \vec{k} = B_R \vec{e}_R + B_Z \vec{e}_z + B_\phi \vec{e}_\varphi$$

$$R = \sqrt{x^2 + y^2}$$
$$\phi = \arctan \frac{y}{x}$$
$$z = Z$$

the inverse map is

$$x = R\cos\phi$$
$$y = R\sin\phi$$
$$z = Z$$

The toroidal component of the magnetic field produced by the external coils is

$$B_\phi = B_0 R_0 / R$$

# Symmetries of a Tokamak plasma

Plasma -> toroidal current -> B in (R,Z) plane
From Ampere's low:

$$\nabla \times B = \frac{4\pi}{c} j \qquad (1)$$

$$j_\phi = j_\phi(R) \text{ for } R \in [R_0 - a, R_0 + a]$$

$$j_\phi = 0 \text{ for } |R\text{-}R_0| > a$$

$R_0$ : centre of the vacuum vessel.
Toroidal component of eq (1) is:

$$\left( \frac{\partial B_Z}{\partial R} - \frac{\partial B_R}{\partial Z} \right) = \frac{4\pi}{c} j_\phi(R) \qquad (2)$$

Use also divergence of B equal 0

$$\left( \frac{1}{R} \frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial Z} \right) = 0 \qquad (3)$$

# Symmetries of a Tokamak plasma

Let us take

$$j_\phi(R) = C(R+R_0)/R^2$$

in this case the solution of equations (2) and (3) is

$$B_R = -\frac{ZB_{p0}}{R}; B_Z = \frac{B_{p0}(R-R_0)}{R}$$

**cylindrical coordinates -> all B components are non zero**

Curves on the R,Z plane that have B as tangent vector

$$\nabla\psi(R,Z)\cdot B = 0$$

$$\psi(R,Z) = const$$

$$\frac{\partial\psi}{\partial R}B_R + \frac{\partial\psi}{\partial Z}B_Z = 0 \Rightarrow \frac{\partial\psi}{\partial R} = f(R,Z)(R-R_0); \frac{\partial\psi}{\partial Z} = f(R,Z)Z$$

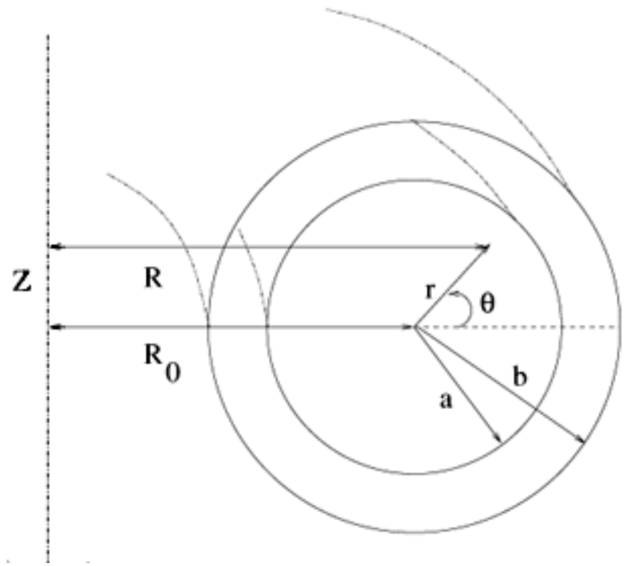
$$\nabla\psi\cdot\nabla\psi = 1 \longrightarrow f(R,Z) = 1/\sqrt{(R-R_0)^2 + Z^2}$$

**family of concentric circles**

$$\psi(R,Z) = \sqrt{(R-R_0)^2 + Z^2}$$

# Symmetries of a Tokamak plasma

**The system of nested toroidal surfaces tangent to the magnetic field (magnetic surfaces)**




The toroidal magnetic field of the space (x,y,z) can be written in a very simple way if we introduce the new system of coordinates

**( $\psi$ in the figure has to be identified with *r*):**

$$\psi = \sqrt{(\sqrt{x^2 + y^2} - R_0)^2 + Z^2}$$

$$\theta = \arctan \frac{Z}{\sqrt{x^2 + y^2} - R_0} \qquad (4)$$

$$\phi = \arctan \frac{y}{x}$$

**and the inverse map**

$$x = \left(R_0 + \psi \cos\theta\right)\cos\phi$$

$$y = \left(R_0 + \psi \cos\theta\right)\sin\phi \qquad (5)$$

$$z = \psi \sin\theta$$

# Symmetries of a Tokamak plasma

In this system of coordinates a function

$$f = f(\psi) = const$$

family of nested toroidal surfaces $R_0, \psi$

**vector fields have components**

$$B = B_x \vec{i} + B_y \vec{j} + B_z \vec{k} = B_R \vec{e}_R + B_Z \vec{e}_z + B_\phi \vec{e}_\phi = B_\psi \vec{e}_\psi + B_\theta \vec{e}_\theta + B_\phi \vec{e}_\phi$$

**relation between the components**

$$B_\psi = \frac{R - R_0}{\sqrt{(R-R_0)^2 + Z^2}} B_R + \frac{Z}{\sqrt{(R-R_0)^2 + Z^2}} B_Z$$

$$B_\theta = -\frac{Z}{\sqrt{(R-R_0)^2 + Z^2}} B_R + \frac{R - R_0}{\sqrt{(R-R_0)^2 + Z^2}} B_Z$$

$$B_\phi = B_\phi$$

**divergence**

$$\nabla \cdot B = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_Z}{\partial Z} + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi}$$

$$\nabla \cdot B = \frac{1}{\psi}\frac{\partial}{\partial \psi}(\psi B_\psi) + \frac{1}{\psi}\frac{\partial B_\theta}{\partial \theta} + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} + \frac{B_\psi \cos\theta}{R} - \frac{B_\theta \sin\theta}{R}$$

# Symmetries of a Tokamak plasma

The magnetic field in toroidal coordinates is:

$$B_\psi = 0; \quad B_\vartheta = \frac{B_{p0}\psi}{R_0 + \psi\cos\theta}; B_{\phi 0} = \frac{B_0 R_0}{R_0 + \psi\cos\theta}$$

$B_{\phi 0}$ -> vacum toroidal magnetic field

**it is easy to show that the**

$$\nabla \cdot B = 0$$

**for any**

$$B_\vartheta = \frac{B_{\theta 0}(\psi) R_0}{R_0 + \psi\cos\theta}$$

general case: current of equation (2) more complicated function of $R$ -> **the magnetic surfaces X-section are not concentric circles**

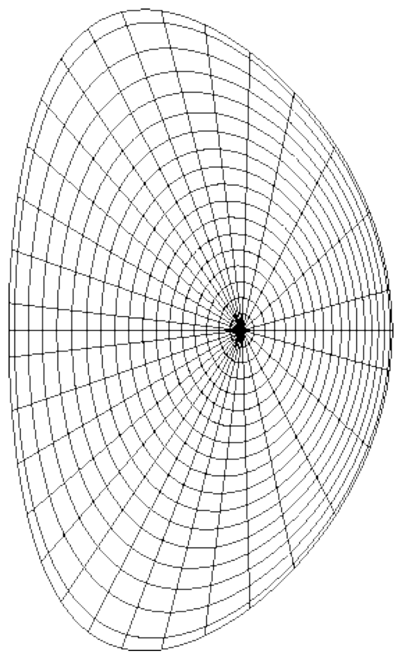
$\psi$ in general will not be the radius of the magnetic surface determined by

$$\nabla\psi(R,Z)\cdot B = 0$$

$$\frac{\partial\psi}{\partial R} = -B_Z; \frac{\partial\psi}{\partial Z} = B_R \Rightarrow \left|\nabla\psi\right| = B_p$$

# Symmetries of a Tokamak plasma

An example of realistic tokamak magnetic surfaces arising from the balance between the pressure gradient and the jxB force (Grad-Shafranov equation) is given by the function:

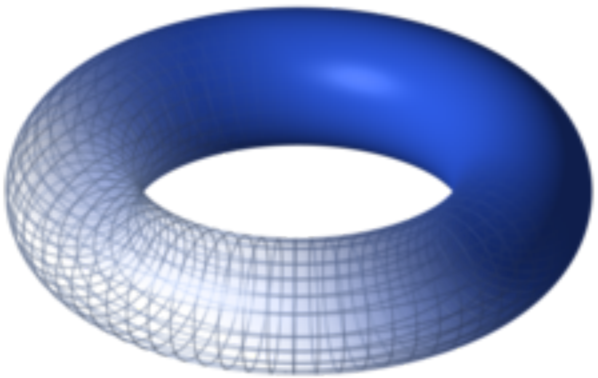


$$\psi(R,Z) = \frac{\psi_0}{R_0^4}\left[\left(R^2 - R_0^2\right)^2 + \frac{Z^2}{E^2}\left(R^2 - R_x^2\right) - \tau R_0^2\left(R^2 \ln\frac{R^2}{R_0^2} - \left(R^2 - R_0^2\right) - \frac{\left(R^2 - R_0^2\right)^2}{2R_0^2}\right)\right]$$

# Volume Element and Surface Average

Let $f(\psi,\phi,\theta)$ be a function of $\Re \rightarrow \Re$ and let be $\psi(R,Z) = cons$

be a torus in the (R,Z) cylindrical space.



**We define**:

$$\langle f(\psi,\phi,\theta)\rangle \equiv \lim_{\delta\to 0}\frac{\int_{\psi-\delta}^{\psi+\delta}\delta\left(\overline{\psi}-\psi\right)\int_0^{2\pi}\int_0^{2\pi} f\left(\overline{\psi},\theta,\phi\right)\left|J\right| d\overline{\psi}\, d\theta\, d\phi}{\int_{\psi-\delta}^{\psi+\delta}\delta\left(\overline{\psi}-\psi\right)\int_0^{2\pi}\int_0^{2\pi} J\, d\overline{\psi}\, d\theta\, d\phi} = \overline{f}(\psi)$$

# Volume Element and Surface Average

J -> determinant of the Jacobian

**J depends on**

$$\psi(R,Z) \qquad \theta(R,Z)$$

we will take the map (4) and (5) and calculate the determinant of the Jacobian of map (4).

**by definition**

$$dV = dxdydz = Jd\psi d\theta d\phi$$

**where**

$$J \equiv \frac{\partial(x,y,z)}{\partial(\psi,\theta,\phi)} \qquad \textbf{and} \qquad 1/J \equiv \frac{\partial(\psi,\theta,\phi)}{\partial(x,y,z)}$$

# Volume Element and Surface Average

$$\frac{\partial(\psi,\theta,\phi)}{\partial(x,y,z)} = \det\begin{vmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \qquad \frac{\partial(x,y,z)}{\partial(\psi,\theta,\phi)} = \det\begin{vmatrix} \frac{\partial x}{\partial\psi} & \frac{\partial y}{\partial\psi} & \frac{\partial z}{\partial\psi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

**from the map (5) it is to see that**

$$\frac{\partial(x,y,z)}{\partial(\psi,\theta,\phi)} = \det\begin{vmatrix} \frac{\partial x}{\partial\psi} = \cos\theta\cos\phi & \frac{\partial y}{\partial\psi} = -\psi\sin\theta\cos\phi & \frac{\partial z}{\partial\psi} = -(R_0+\psi\cos\theta)\sin\phi \\ \frac{\partial x}{\partial\theta} = \cos\theta\sin\phi & \frac{\partial y}{\partial\theta} = -\psi\sin\theta\sin\phi & \frac{\partial z}{\partial\theta} = (R_0+\psi\cos\theta)\cos\phi \\ \frac{\partial x}{\partial\phi} = \sin\theta & \frac{\partial y}{\partial\phi} = \psi\cos\theta & \frac{\partial z}{\partial\phi} = 0 \end{vmatrix}$$

$$= \begin{matrix} -(R_0+\psi\cos\theta)\sin\phi\left[\psi\cos^2\theta\sin\phi + \psi\sin^2\theta\sin\phi\right] + \\ -(R_0+\psi\cos\theta)\cos\phi\left[\psi\cos^2\theta\cos\phi + \psi\sin^2\theta\cos\phi\right] \end{matrix}$$

$$= -(R_0+\psi\cos\theta)\left[\psi\cos^2\theta + \psi\sin^2\theta\right]$$

$$= -\psi(R_0+\psi\cos\theta)$$

# Volume Element and Surface Average

Therefore for the map (4),(5) we have the result $\left|J\right| = \psi(R_0 + \psi\cos\theta)$

The area of the magnetic surface $\psi$ is promptly calculated:

$$A(\psi) \equiv \lim_{\delta\to 0}\int_{\psi-\delta}^{\psi+\delta}\delta\left(\overline{\psi}-\psi\right)\int_0^{2\pi}\int_0^{2\pi} J d\overline{\psi}\, d\theta\, d\phi$$

$$= 2\pi\psi\int_0^{2\pi}(R_0 + \psi\cos\theta)d\theta = 4\pi^2 R_0\psi$$

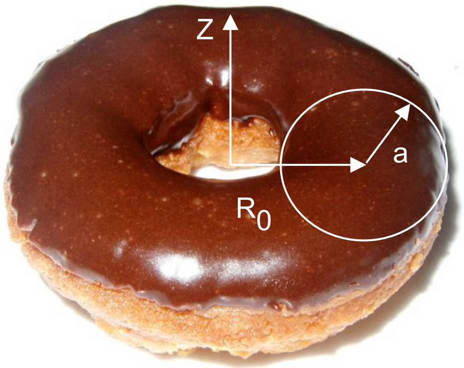
Using the expression of the Jacobian we can write the average of $f(\psi,\phi,\theta)$

on a circular cross section magnetic surface as:

$$\left\langle f(\psi,\phi,\theta)\right\rangle \equiv \frac{\int_0^{2\pi} f(\psi,\phi,\theta)(R_0 + \psi\cos\theta)d\theta}{2\pi R_0} = \overline{f}(\psi) \qquad (6)$$

# calculation of the surface average

As an example of application of equation (6) we will consider the most well known torus: the doughnut




Equation (6) gives

$$\bar{n}(\psi) = \langle n(\psi,\phi,\theta)\rangle \equiv \frac{\int_0^{\pi} n_{c0}(R_0 + \psi\cos\theta)d\theta}{2\pi R_0} = \pi\frac{n_{c0}R_0}{2\pi R_0} + \frac{n_{c0}\psi}{2\pi R_0}[\sin\theta]_0^{\pi} = \frac{n_{c0}}{2}$$

**The second term in the sum above does not contribute due to the symmetry of the chocolate distribution around** $\theta = \pi/2$

We aim at calculating the average density of chocolate on the surface of a doughnut of major radius $R_0$ and minor radius

$$n_c(\psi,\phi,\theta)$$

is the density of the chocolate on the surface

A different result Is obtained if we consider the chocolate to be localized on the outer side of the doughnut surface (**considerably more difficult to achieve in practice**!!!)

$$n_c(\psi,\phi,\theta) = \begin{cases} n_{c0} & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$$

$$n_c(\psi,\phi,\theta) = \begin{cases} n_{c0} & 0 < \theta < \frac{\pi}{2};\quad \frac{3}{2}\pi < \theta < 2\pi \\ 0 & \frac{\pi}{2} < \theta < \frac{3}{2}\pi \end{cases}$$

# calculation of the surface average

$$\bar{n}(\psi) = \langle n(\psi,\phi,\theta)\rangle \equiv \frac{2\int_0^{\pi/2} n_{c0}(R_0+\psi\cos\theta)d\theta}{2\pi R_0} = 2\frac{1}{2}\pi\frac{n_{c0}R_0}{2\pi R_0} + 2\frac{n_{c0}\psi}{2\pi R_0}\left[\sin\theta\right]_0^{\pi/2} = \frac{n_{c0}}{2} + \frac{n_{c0}\psi}{\pi R_0}$$

This is due to the fact that the outer area of the torus is larger than the inner. For a large aspect ratio doughnut ( $\varepsilon = a/R_0$ ) the difference between the outer area of the doughnut and that of an equivalent straight cylinder (**eclair au chocolat**) is equal to

$$\Delta A^{1/2} = A_T^{1/2} - A_C^{1/2} = \frac{\varepsilon}{\pi}$$

Now coming back to the tokamak problem, we observe that the average value of the toroidal magnetic field on the magnetic surface is equal to the magnetic field at the surface axis.

$$< B_\phi > = < \frac{B_0R_0}{R_0+\psi\cos\theta} > = \frac{\int_0^{2\pi}\frac{B_0R_0}{R_0+\psi\cos\theta}(R_0+\psi\cos\theta)d\theta}{2\pi R_0} = B_0$$

# Introduction of the transport problem

Let us consider now a tokamak plasma described by the electron and ion density and pressure

$$n_{e,i}(\psi,\theta) \qquad p_{e,i}(\psi,\theta)$$

Let us consider now the toroidal volume enclosed in the toroidal surface $\psi(R,Z)$

**The total number of particles and pressure inside the volume is**

$$N_{e,i} = \int_V n_{e,i}(\psi,\theta)dV \qquad P_{e,i} = \int_V p_{e,i}(\psi,\theta)dV$$

The goal of transport theory is to determine the confinement time defined as:

$$\frac{\partial N_{e,i}}{\partial t} = -\frac{N_{e,i}}{\tau^n_{e,i}} \quad ; \quad \frac{\partial P_{e,i}}{\partial t} = -\frac{P_{e,i}}{\tau^p_{e,i}}$$

the mass continuity equation, in the absence of sources, is

$$\frac{\partial n_{e,i}}{\partial t} = -\nabla \cdot n_{e,i} V_{e,i}$$ , and by taking the volume integral

# Introduction of the transport problem

$$\frac{\partial N_{e,i}}{\partial t} = -\int_V \nabla \cdot n_{e,i} V_{e,i} dV = -\int_S n_{e,i} V_{e,i} \cdot dS$$

the unit vector normal to the circular magnetic surface has radial direction and the surface element has been calculated before therefore we can write (using eq. 6)

$$\frac{\partial N_{e,i}}{\partial t} = -2\pi\psi \int_0^{2\pi} n_{e,i} V_{e,i\psi} (R_0 + \psi \cos\theta) d\theta = -\left\langle n_{e,i} V_{e,i\psi} \right\rangle A$$

**The goal of transport theory in the plasma core is to calculate**

$$\left\langle n_{e,i} V_{e,i\psi} \right\rangle \quad \text{and} \quad \left\langle n_{e,i} T_{e,i} V_{e,i\psi} \right\rangle$$

# Classical and Neoclassical component

The radial flux of particles and heat has to be calculated from the momentum and energy conservation

$$m_i n_i \frac{\partial v_i}{\partial t} + m_i n_i v_i \cdot \nabla v_i = -\nabla p_i + \nabla \cdot \pi_i + n_i e\left(E + v_i \times B\right) - F_{ei}$$

$$m_e n_e \frac{\partial v_e}{\partial t} + m_e n_e v_e \cdot \nabla v_e = -\nabla p_e + \nabla \cdot \pi_e - n_e e\left(E + v_e \times B\right) + F_{ei}$$

$$\frac{3n_i}{2}\left(\frac{\partial}{\partial t} + v_i \cdot \nabla\right) T_i + p_i \nabla \cdot v_i = -\nabla \cdot q_i - \pi_i : \nabla v_i + Q_i$$

$$\frac{3n_e}{2}\left(\frac{\partial}{\partial t} + v_e \cdot \nabla\right) T_e + p_e \nabla \cdot v_{ei} = -\nabla \cdot q_e - \pi_e : \nabla v_e - Q_e$$

We will calculate the radial flux of particles and heat in the limit of short parallel mean free path ($V_{th}/\nu < Rq$), which implies high collisionality ($\nu^* = \nu Rq / V_{th} >> 1$)

In the above limit the friction term and heat flux have been calculated by Braginskii [1] and are

$$F_{ei} = -m_e n_e \nu_{ei}\left(C_1 u_{\|} + u_{\perp}\right) - C_2 n \nabla_{\|} T_e - \frac{3}{2}\frac{n}{\omega_{ce}} \nu_{ei} b \times \nabla T_e$$

$$q_e = nT_e\left(C_2 u_{\|} + \frac{3/2}{\omega_{ce}} \nu_{ei} b \times u\right) - \frac{n_e T_{ei}}{m_e \nu_{ei}}\left(C_3 \nabla_{\|} T_e + \frac{C_4}{\omega^2{}_{ce}} \nu^2{}_{ei} \nabla T_e - \frac{5/2}{\omega_{ce}} \nu_{ei} b \times \nabla T_e\right) \qquad (7)$$

# Classical and Neoclassical Component

Calculate the velocity perpendicular to the magnetic field by taking the xB product:

$$v_{i\perp} = \frac{\nabla p_i \times B}{n_i e B^2} - \frac{E \times B}{B^2} + \frac{F_{ei} \times B}{n_i e B^2}$$

$$v_{e\perp} = -\frac{\nabla p_e \times B}{n_e e B^2} - \frac{E \times B}{B^2} + \frac{F_{ei} \times B}{n_e e B^2} \qquad (8)$$

**where**

$$v_\perp = v - (v \cdot b) b = v_\psi e_\psi + \left( v_\theta \frac{B_\phi}{B} - v_\phi \frac{B_\theta}{B} \right) \frac{B_\phi}{B} e_\theta + \left( v_\phi \frac{B_\theta}{B} - v_\theta \frac{B_\phi}{B} \right) \frac{B_\theta}{B} e_\phi$$

The product on the right hand side gives the following components
(assuming toroidal symmetry and taking field components of tokamak equilibrium magnetic field).

$$\frac{\nabla p \times B}{neB^2} = \frac{1/\psi \partial_\theta p B_\phi}{neB^2} e_\psi - \frac{\partial_\psi p B_\phi}{neB^2} e_\theta + \frac{\partial_\psi p B_\theta}{neB^2} e_\phi$$

$$\frac{E \times B}{B^2} = \left( \frac{E_\theta B_\phi}{B^2} - \frac{E_\phi^A B_\theta}{B^2} \right) e_\psi - \frac{E_\psi B_\phi}{B^2} e_\theta + \frac{E_\psi B_\theta}{B^2} e_\phi \qquad (9)$$

$$\frac{F_{ei} \times B}{neB^2} = \left( \frac{F_{ei\theta} B_\phi}{neB^2} - \frac{F_{ei\phi} B_\theta}{neB^2} \right) e_\psi - \frac{F_{ei\psi} B_\phi}{neB^2} e_\theta + \frac{F_{ei\psi} B_\theta}{neB^2} e_\phi$$

# Classical and Neoclassical Component

The local radial (perpendicular to the magnetic surface) ion flux is (note that $n_i v_{i\psi} = n_e v_{e\psi}$ )

$$n_i v_{i\psi} = \frac{1/\psi \partial_\theta p_i B_\phi}{eB^2} - \frac{n_i E_\theta B_\phi}{B^2} + \frac{n_i E_\phi^A B_\theta}{B^2} + \frac{F_{ei\theta} B_\phi}{eB^2} - \frac{F_{ei\phi} B_\theta}{eB^2} =$$

$$= \frac{B_\phi}{eB^2}\left(\frac{1}{\psi}\frac{\partial p_i}{\partial \theta} - n_i e E_\theta\right) + \frac{F_{ei\theta} B_\phi}{eB^2} - \frac{F_{ei\phi} B_\theta}{eB^2} + \frac{n_i e E_\phi^A B_\theta}{B_\phi} \qquad (10)$$

$$\Gamma_i = \left\langle n_i v_{i\psi} \right\rangle = \left\langle \frac{B_\phi}{eB^2}\left(\frac{1}{\psi}\frac{\partial p_i}{\partial \theta} - n_i e E_\theta\right)\right\rangle - \left\langle \frac{F_{ei\perp\phi}}{eB_\theta}\right\rangle + \left\langle \frac{n_i E_\phi^A B_\theta}{B^2}\right\rangle = \Gamma_i^{PS} + \Gamma_i^{cl} + \Gamma_i^{E^A \times B}$$

The flux arising from the **toroidal component of the perpendicular friction** is due to the diamagnetic current and is the basic collisional flux that is present also in cylindrical geometry. The toroidal electric field is fully inductive.

The flux arising from the poloidal component of the pressure gradient and electric field is a neoclassical effect due to the non uniformity of the magnetic field over the magnetic surface. It can be seen as due to the friction acting on the gyro centres. **We will show later that this term is proportional to the** toroidal component of the friction.

# Calculation of the classical ion flux

Using equation (7) and taking the toroidal component

$$F_{ei\perp} = -m_e n_e \nu_{ei} (u_\perp)$$

Now substituting the perpendicular current from (8), (9)

$$u_{\perp\varphi} = \frac{\partial_\psi p_i B_\theta}{n_i e B^2} + \frac{\partial_\psi p_e B_\theta}{n_e e B^2}$$

substituting in (10) we get

$$\Gamma_i^{cl} = \left\langle n_i v_{i\psi} \right\rangle = -\left\langle \frac{F_{ei\perp\phi}}{eB_\theta} \right\rangle = \left\langle \frac{m_e n_e \nu_{ei}}{e^2 B^2} \left( \frac{\partial_\psi p_i}{n_i} + \frac{\partial_\psi p_e}{n_e} \right) \right\rangle$$

at the zero order in epsilon, we have

$$\Gamma_i^{cl} = -\frac{m_i n_i \nu_{ie}}{e^2 B^2} \left\langle \left( \frac{T_i \partial_\psi n_i}{n_i} + \frac{T_e \partial_\psi n_e}{n_e} + \frac{n_i \partial_\psi T_i}{n_i} + \frac{n_e \partial_\psi T_e}{n_e} \right) \right\rangle =$$

$$= -\frac{m_i T_i \nu_{ie}}{e^2 B^2} \left( \partial_\psi n_i + \frac{n_i T_e \partial_\psi n_e}{T_i n_e} + \frac{n_i \partial_\psi T_i}{T_i} + \frac{n_i \partial_\psi T_e}{T_i} \right)$$

# Calculation of the classical ion flux

By replacing $\rho_i^2 = \dfrac{m_i T_i}{e^2 B^2}$ therefore we can write the classical flux as

$$\Gamma_i^{cl} = -\rho^2 \nu_{ie}\left(\partial_\psi n_i + \frac{n_i T_e \partial_\psi n_e}{T_i n_e} + \frac{n_i \partial_\psi T_i}{T_i} + \frac{n_i \partial_\psi T_e}{T_i}\right) \qquad (12)$$

In a two species plasma the ion density equals the electron density and expression (12) becomes

$$\Gamma_i^{cl} = -\rho^2 \nu_{ie}\left(\left(1 + \frac{T_e}{T_i}\right)\partial_\psi n + n\left(\frac{\partial_\psi T_i}{T_i} + \frac{\partial_\psi T_e}{T_i}\right)\right)$$

•The diffusion coefficient is defined as the coefficient which multiplies the density gradient of the transported species.

•The classical diffusion coefficient is that of ions performing a random walk in the radial direction with step the ion Larmor radius and frequency the ion electron collision frequency.

•In the derivation of the classical flux we have retained only term of zero order in epsilon.

# Calculation of the P-S particle and heat flux

In toroidal geometry the perpendicular current is not divergence free

$$j_\perp = ne\left(v_{e\perp} - v_{i\perp}\right) = -ne\left(\frac{\nabla p_e \times B}{n_e e B^2} + \frac{\nabla p_i \times B}{n_i e B^2}\right)$$

$$\nabla \cdot j = \frac{1}{\psi}\frac{\partial}{\partial \psi}\left(\psi j_\psi\right) + \frac{1}{\psi}\frac{\partial j_\theta}{\partial \theta} + \frac{1}{R}\frac{\partial j_\phi}{\partial \phi} + \frac{j_\psi \cos\theta}{R} - \frac{j_\theta \sin\theta}{R}$$

Since the radial component of the perpendicular current is zero for ambipolarity and toroidal symmetry

Only the theta components of the perpendicular current enters the divergence

$$j_{\perp\theta} = ne\left(\frac{\partial_\psi p B_\phi}{neB^2} + \frac{\partial_\psi p B_\phi}{neB^2}\right)$$

$$\nabla \cdot j_\perp = \frac{1}{\psi}\frac{\partial}{\partial \theta}\left(\frac{\partial_\psi p_e B_\phi}{B^2} + \frac{\partial_\psi p_i B_\phi}{B^2}\right) - \frac{\sin\theta}{R}\left(\frac{\partial_\psi p_e B_\phi}{B^2} + \frac{\partial_\psi p_i B_\phi}{B^2}\right)$$

$$\frac{\partial}{\partial \theta}\left(\frac{B_\phi}{B^2}\right) \cong \frac{\partial}{\partial \theta}\left(\frac{1}{B_\phi}\right) = \frac{\partial}{\partial \theta}\left(\frac{R_0 + \psi\cos\theta}{B_0}\right) = -\frac{\psi}{B_0}\sin\theta$$

$$\nabla \cdot j_\perp = -\frac{1}{B_0}\left(\partial_\psi p_e + \partial_\psi p_i\right)\sin\theta - \frac{\sin\theta}{R}\left(\frac{\partial_\psi p_e B_\phi}{B^2} + \frac{\partial_\psi p_i B_\phi}{B^2}\right)$$

$$\nabla \cdot j_\perp = \nabla \cdot j_{\perp\theta} = -\frac{2}{B_0}\left(\partial_\psi p_e + \partial_\psi p_i\right)\sin\theta$$

# Calculation of the P-S particle and heat flux

For charge conservation

$$\nabla\cdot\left(j_{\parallel\theta}+j_{\perp\theta}\right)=0$$

The poloidal component of the parallel current is calculated below

$$j_{\parallel}=(j\cdot b)b=\left(j_\theta\frac{B_\theta}{B}+j_\phi\frac{B_\phi}{B}\right)\frac{B_\theta}{B}e_\theta+\left(j_\theta\frac{B_\theta}{B}+j_\phi\frac{B_\phi}{B}\right)\frac{B_\phi}{B}e_\phi$$

Therefore

$$j_{\parallel\theta}=\left(j_\theta\frac{B_\theta}{B}+j_\phi\frac{B_\phi}{B}\right)\frac{B_\theta}{B}$$

The condition for divergence free current is

$$j_{\parallel\theta}+j_{\perp\theta}=j_\theta=\frac{j_{\theta 0}(\psi)}{1+\varepsilon\cos\theta}$$

which is an equation for the toroidal component of the current

$$j_\theta\frac{B_\theta^2}{B^2}+j_\phi\frac{B_\theta B_\phi}{B^2}+j_{\perp\theta}=\frac{j_{\theta 0}(\psi)}{1+\varepsilon\cos\theta}$$

# Calculation of the P-S particle and heat flux

Adding the poloidal components we get

$$j_\phi \frac{B_\theta B_\phi}{B^2} + j_{\perp\theta} = \frac{j_{\theta 0}(\psi)}{1+\varepsilon\cos\theta}\frac{B_\phi^2}{B^2}$$

At zero order (in epsilon) the poloidal current is the diamagnetic current plus the inductive current (from radial component of $j \times B = \nabla(p_e + p_i)$ ) $\quad j_\phi = j_{\phi ohm} + j_{\phi 1}$

$$j_{\theta 0}(\psi) = j_{\phi ohm}\frac{B_{\theta 0}}{B_{\phi 0}} + \frac{1}{B_{\phi 0}}\left(\partial_\psi p_i + \partial_\psi p_e\right)$$

substituting in the equation together with the poloidal component of the perpendicular current at first order in epsilon

$$-\frac{\varepsilon\cos\theta}{B_{\phi 0}}\left(\partial_r p_i + \partial_r p_e\right) - j_{\phi ohm}\varepsilon\cos\theta\frac{B_{\theta 0}}{B_{\phi 0}} = j_{\phi 1}\frac{B_{\theta 0}}{B_{\phi 0}} + \frac{\varepsilon\cos\theta}{B_{\phi 0}}\left(\partial_r p_i + \partial_r p_e\right)$$

$$j_{\phi 1} = -j_{\phi ohm}\varepsilon\cos\theta - \frac{2\varepsilon\cos\theta}{B_{\theta 0}}\left(\partial_r p_i + \partial_r p_e\right)$$

finally

$$j_\phi = j_{\phi ohm}(1-\varepsilon\cos\theta) - \frac{2\varepsilon\cos\theta}{B_{\theta 0}}\left(\partial_r p_i + \partial_r p_e\right) = j^A + j^{ps}$$

# Calculation of the P-S particle and heat flux

We will use the PS current to compute the neoclassical radial flux

From the parallel component of the momentum balance equation for ions and electrons in the PS regime of collisionality

$$0 = -\nabla p_i + ne\left(E + v_i \times B\right) - F_{ei}$$

$$0 = -\nabla p_e - ne\left(E + v_e \times B\right) + F_{ei}$$

The parallel component is (*where the F contain only the non inductive parallel current*):

$$\frac{B}{B_\theta}\overline{F}_{ei\parallel} = -\frac{1}{\psi}\frac{\partial p_i}{\partial \theta} + neE_\theta$$

$$\frac{B}{B_\theta}\overline{F}_{ei\parallel} = \frac{1}{\psi}\frac{\partial p_e}{\partial \theta} + neE_\theta$$

From equation (10) we have

$$\Gamma^{PS} = -\left\langle \frac{B_\phi}{eB^2}\left(\frac{1}{\psi}\frac{\partial p_e}{\partial \theta} + neE_\theta\right)\right\rangle = -\left\langle \frac{\overline{F}_{ei\parallel\phi}}{eB_\theta}\right\rangle$$

**The PS flux arises from averaging over the magnetic surface the toroidal component of the parallel friction (non inductive parallel current).**

$$\overline{F}_{ei\parallel} = -m_e n \nu_{ei} C_1\left(u_\parallel - u_\parallel^A\right) - C_2 n \frac{1}{\psi}\frac{\partial T_e}{\partial \theta}\frac{B_\theta}{B}$$

and using $u_\parallel - u_\parallel^A \approx \dfrac{j_\phi^{PS}}{ne}$

we have

$$-m_e \nu_{ei} C_1 \frac{B}{B_\theta}\frac{j_\phi^{PS}}{e} - C_2 n \frac{1}{\psi}\frac{\partial T_e}{\partial \theta} = \frac{1}{\psi}\frac{\partial p_e}{\partial \theta} + neE_\theta$$

$$m_e \nu_{ei} C_1 \frac{B}{B_\theta}\frac{2\varepsilon\cos\theta}{eB_{\theta 0}}\left(\partial_\psi p_e + \partial_\psi p_i\right) - C_2 n \frac{1}{\psi}\frac{\partial T_e}{\partial \theta} = \frac{1}{\psi}\frac{\partial p_e}{\partial \theta} + neE_\theta$$

Need to find poloidal temperature gradient

# Calculation of the P-S particle and heat flux

Following Rutherford (PF, 1974) we use the heat flux and calculate the toroidal component of the heat flux from taking the leading component of the perpendicular flux within the magnetic surface:

$$q_{e\perp} = \frac{5n_eT_e}{2}\left(\frac{1}{eB}b\times\nabla T_e\right)$$

The poloidal component is

$$q_{e\theta} = \frac{B_\theta}{B_\phi}q_{e\phi} + \frac{5n_eT_e}{2eB}\frac{\partial T_e}{\partial r}$$

By assuming that the heat flow is divergence free to first order in epsilon we have

$$q_{e\phi} = -\frac{5\varepsilon nT_e\cos\theta}{eB_{\theta 0}}\frac{\partial T_e}{\partial \psi}$$

# Calculation of the P-S particle and heat flux

taking the parallel component of the heat flux

$$q_{e\parallel} = nT_e\left(C_2 u_\parallel\right) - \frac{nT_e}{m_e\nu_{ei}}\left(C_3\nabla_\parallel T_e\right)$$

Substituting the toroidal component of q and the PS current in place of the parallel velocity

$$-\frac{B}{B_\theta}\frac{5\varepsilon nT_e\cos\theta}{eB_{\theta 0}}\frac{\partial T_e}{\partial\psi} = -T_eC_2\frac{B}{B_\theta}\frac{2\varepsilon\cos\theta}{eB_{\theta 0}}\left(\partial_\psi p_e + \partial_\psi p_i\right) - \frac{nT_e}{m_e\nu_{ei}}C_3\frac{1}{\psi}\frac{\partial T_e}{\partial\theta}$$

$$\frac{B}{B_\theta}\frac{2m_e\nu_{ei}\varepsilon\cos\theta}{C_3eB_{\theta 0}}\left[-n\frac{5}{2}\frac{\partial T_e}{\partial\psi} + C_2\left(\partial_\psi p_e + \partial_\psi p_i\right)\right] = -n\frac{1}{\psi}\frac{\partial T_e}{\partial\theta}$$

Placing the poloidal temperature gradient in the equation for the poloidal pressure gradient we have

$$m_e\nu_{ei}C_1\frac{B}{B_\theta}\frac{2\varepsilon\cos\theta}{eB_{\theta 0}}\left(\partial_\psi p_e + \partial_\psi p_i\right) +$$

$$C_2\frac{B}{B_\theta}\frac{2m_e\nu_{ei}\varepsilon\cos\theta}{C_3eB_{\theta 0}}\left[-n\frac{5}{2}\frac{\partial T_e}{\partial\psi} + C_2\left(\partial_\psi p_e + \partial_\psi p_i\right)\right] = \frac{1}{\psi}\frac{\partial p_e}{\partial\theta} + neE_\theta$$

# Calculation of the P-S particle and heat flux

$$\frac{B_{\phi 0} 2 m_e \nu_{ei} \varepsilon \cos\theta}{e B^2{}_{\theta 0}} \left[ \left( C_1 + \frac{C_2^2}{C_3} \right) \left( \partial_\psi p_e + \partial_\psi p_i \right) - \frac{5}{2} \frac{C_2}{C_3} \frac{\partial T_e}{\partial \psi} \right] = \frac{1}{\psi} \frac{\partial p_e}{\partial \theta} + n e E_\theta$$

Inserting this result into the formula for the PS flux

$$\Gamma^{PS} = -\left\langle \frac{B_\phi}{e B^2} \left( \frac{1}{\psi} \frac{\partial p_e}{\partial \theta} + n e E_\theta \right) \right\rangle$$

$$\Gamma^{PS} = \left\langle -\frac{2 m_e \nu_{ei} \varepsilon \cos\theta}{e B^2{}_{\theta 0}} (1 + \varepsilon \cos\theta) \left[ \left( C_1 + \frac{C_2^2}{C_3} \right) \left( \partial_\psi p_e + \partial_\psi p_i \right) - \frac{5}{2} n \frac{C_2}{C_3} \frac{\partial T_e}{\partial \psi} \right] \right\rangle$$

Now taking the average we get:

$$\Gamma^{PS} = \left\langle -\frac{2 m_e \nu_{ei} \varepsilon \cos\theta}{e B^2{}_{\theta 0}} (1 + \varepsilon \cos\theta) \left[ \left( C_1 + \frac{C_2^2}{C_3} \right) \left( \partial_\psi p_e + \partial_\psi p_i \right) - \frac{5}{2} n \frac{C_2}{C_3} \frac{\partial T_e}{\partial \psi} \right] \right\rangle$$

$$\Gamma^{PS} = -\frac{2 m_e \nu_{ei} \varepsilon^2 B_0^2}{B_0^2 e B^2{}_{\theta 0}} \left[ \left( C_1 + \frac{C_2^2}{C_3} \right) \left( \partial_\psi p_e + \partial_\psi p_i \right) - \frac{5}{2} n \frac{C_2}{C_3} \frac{\partial T_e}{\partial \psi} \right]$$

Splitting the pressure gradient into temperature and density gradient we can write

$$\Gamma^{PS} = -\frac{2 m_e T_e \nu_{ei} \varepsilon^2 B_0^2}{B_0^2 e B^2{}_{\theta 0}} \left[ \left( C_1 + \frac{C_2^2}{C_3} \right) \left( n \frac{1}{T_e} \partial_\psi T_e + n \frac{1}{T_e} \partial_\psi T_i + \frac{T_i}{T_e} \partial_\psi n + \partial_\psi n \right) - \frac{5}{2} \frac{1}{T_e} \frac{C_2}{C_3} n \frac{\partial T_e}{\partial \psi} \right]$$

# Calculation of the P-S particle and heat flux

By identifying $q^2 = \dfrac{\varepsilon^2 B_0^2}{B^2{}_{\theta 0}}$ and $\rho^2 = \dfrac{mT}{B_0^2 e}$ we have

$$\Gamma^{PS} = -2q^2\rho^2\nu\left[\left(C_1 + \frac{C_2^2}{C_3}\right)\left(\left(1+\frac{T_i}{T_e}\right)\partial_\psi n + \frac{n}{T_e}\left(\partial_\psi T_e + \partial_\psi T_i\right)\right) - \frac{5}{2}\frac{n}{T_e}\frac{C_2}{C_3}\frac{\partial T_e}{\partial\psi}\right]$$

As compared to the classical flux, the PS diffusion coefficient and heat conductivity is enhanced by a factor $2q^2$

This enhancement is purely due to the toroidal geometry.

At low collisionality $\nu^* = \nu Rq / V_{th} << 1$

the transport is dominated by particles trapped in the magnetic well (banana particles) the exact calculation of the flux requires the use of the drift Kinetic equation and can be found in e.g.

P. Helander and D.J. Sigmar, Collisional Transport in Magnetized Plasma, Cambridge University Press

The result is that the PS diffusion coefficient and heat conductivity are further enhanced by factor $\varepsilon^{-3/2}$

# Effect of rotation on heavy impurities

From M. Romanelli et al, EPS 1997

$$0 = -\nabla p_e - n_e e(\vec{E} + \vec{V}_e \times \vec{B}) + \vec{F}_{ei} + \vec{F}_{eI} \ ,$$

$$\vec{F}_c^{\,i} = -\nabla p_i + Z_i n_i e(\vec{E} + \vec{V}_i \times \vec{B}) + \vec{F}_{iI} + \vec{F}_{ie} \ , \qquad (1)$$

$$\vec{F}_c^{\,I} = -\nabla p_I + Z_I n_I e(\vec{E} + \vec{V}_I \times \vec{B}) + \vec{F}_{Ii} + \cancel{\vec{F}_{Ie}} \ ,$$

$$\vec{F}_{ij} = -m_i n_i \upsilon_{ij} C_1 \vec{u}_{\|} - C_2 n_i \nabla_{\|} T_i$$

$$\vec{F}_c = -mn\Omega^2 R^2 \vec{e}_R$$

$$\vec{u} = \vec{v}_i - \vec{v}_j \ .$$

Momentum balance equation for electrons and two ion species

# Equilibrium with rotation

From M. Romanelli et al, EPS 1997

$$V^{0}_{\perp\theta I} = \frac{\partial p_I}{\partial r}\frac{1}{n_I e Z_I B_T} - \frac{m_I \Omega^2 (R_0 + r\cos\theta)\cos\theta}{e Z_I B_T} - \frac{E_r}{B_T} \quad . \qquad (2)$$

- Trace impurity approximation -> E is determined by i,e

$$V^{0}_{\perp\theta I} = V^{0}_{\perp\theta i}$$

equilibrium -> no friction i,I

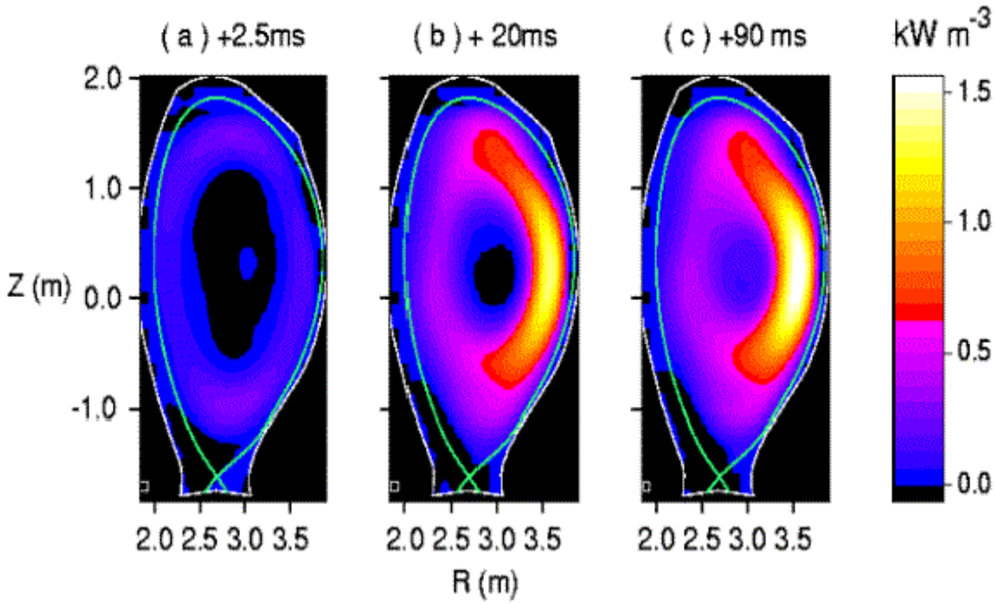
$$n_I = n_I(0)\left(\frac{n_i(r)}{n_i(0)}\right)^{\frac{Z_I}{Z_i}} \exp\left[\frac{\Omega^2}{2T} m_I \left(1 - \frac{m_i}{m_I}\frac{Z_I T_e}{T + Z_i T_e}\right)\left(R^2 - R_0^{\ 2}\right) - \frac{\Omega^2}{2}\frac{Z_I}{Z_i}\frac{m_i}{T + Z_i T_e}\left(r^2 - 2rR_0\right)\right]$$

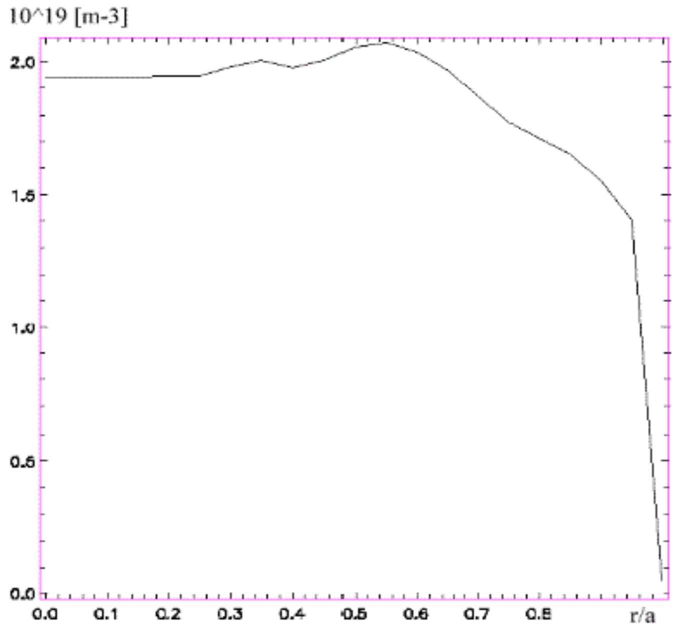
This solution assumes trace impurity!

# Observation of Ni asymmetry in JET

From M. Romanelli et al, EPS 1997

$$M_{i,I} = V_{\phi i,I} / V_{thi,I} \qquad M_i \approx O(\Delta) \qquad M_I \approx O(1)$$




JET Ni LBO experiment (C-wall), 1997

# Effect of Rotation on Impurity transport

In the presence of rotation heavy impurities in the PS regime such as Ni and W have Mach number of order 1 it is show in **M. Romanelli M. Ottaviani, 1998, PPCF** that

$$\Gamma^{PS} = < n_I V_{Ir} >_\psi = D_\Omega \partial_r \langle n_I \rangle_\psi + V_p \langle n_I \rangle_\psi$$

$$D_\Omega = D_{PS}(1 + M^*)^2 = D_{PS}(1 + \frac{m^* \Omega^2 R_0^2}{2T_i})^2$$

$$V_p = D_\Omega Z \frac{\partial_r p_{i0}}{p_{i0}} + D_\Omega P$$

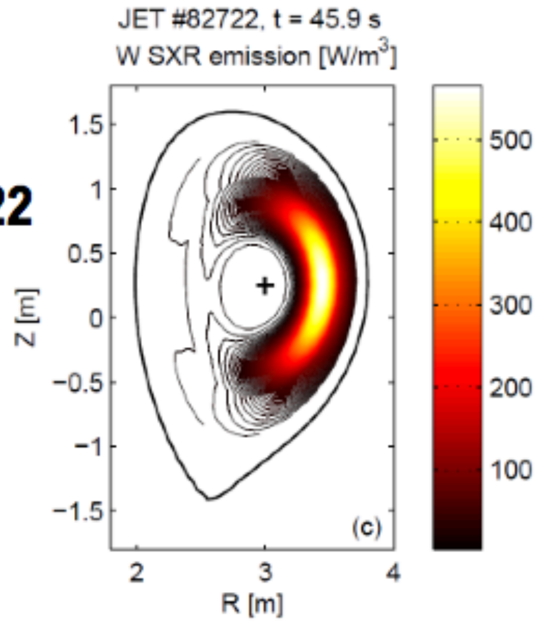
$$P = \frac{\tilde{m}}{m^*}\left( \frac{M^*(1 + 3\varepsilon M^* + 2\varepsilon M^{*2}) - R_0 \partial_r(\varepsilon M^*)}{R_0 \varepsilon (1 + M^*)^2} \right)$$

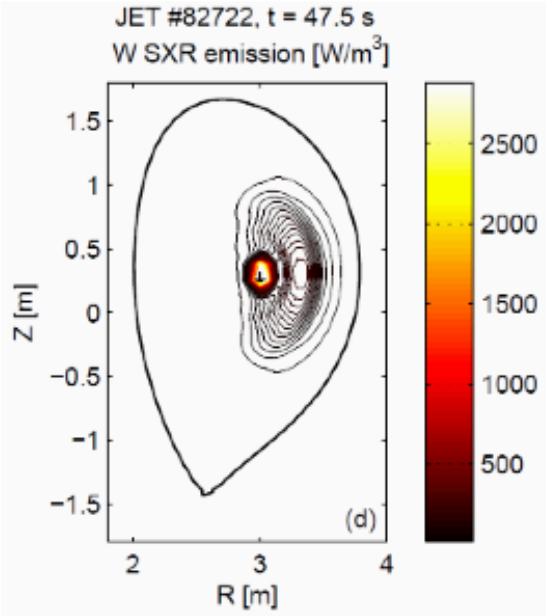
Where $\tilde{m} = m_I - Z m_i$ and $m^* = (m_I - Z m_i T_e / (T_e + T_i))$

# Comparison with W studies in JET

JET #82722

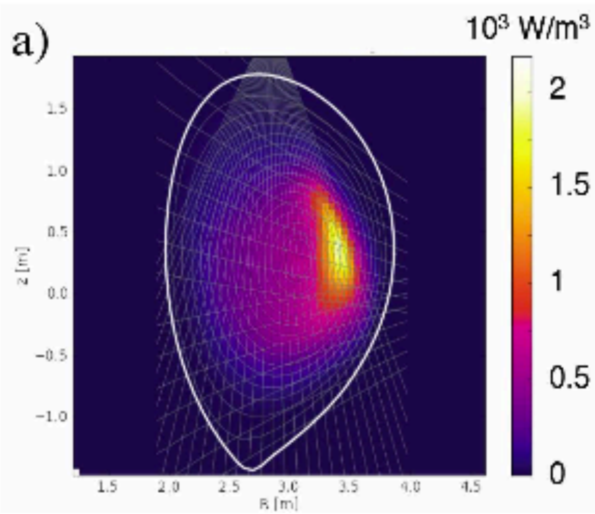
- Predicted SXR power density emitted by W (using W SXR cooling factor)

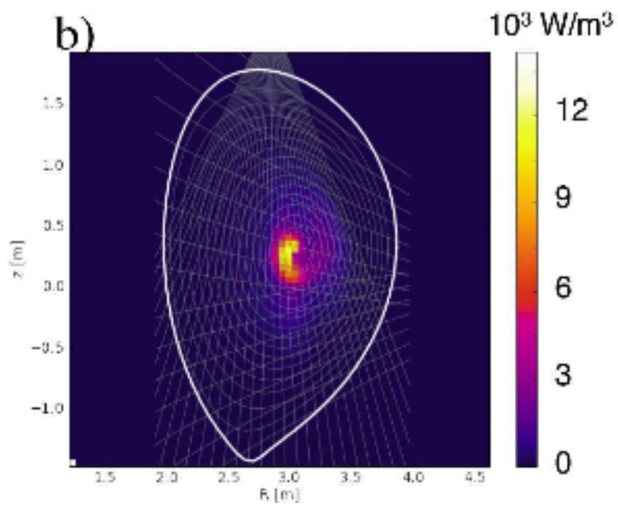



- SXR tomography [courtesy of J. Mlynar]

- Up-down asymmetry might be evidence of impurity-ion friction effect, not included in the modelling

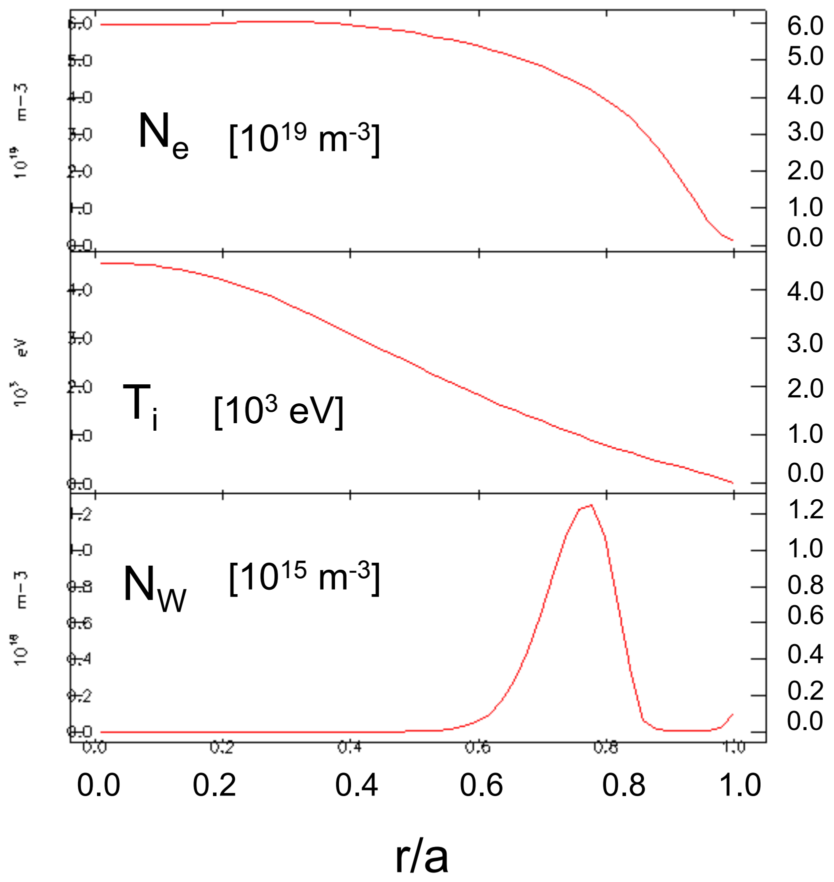


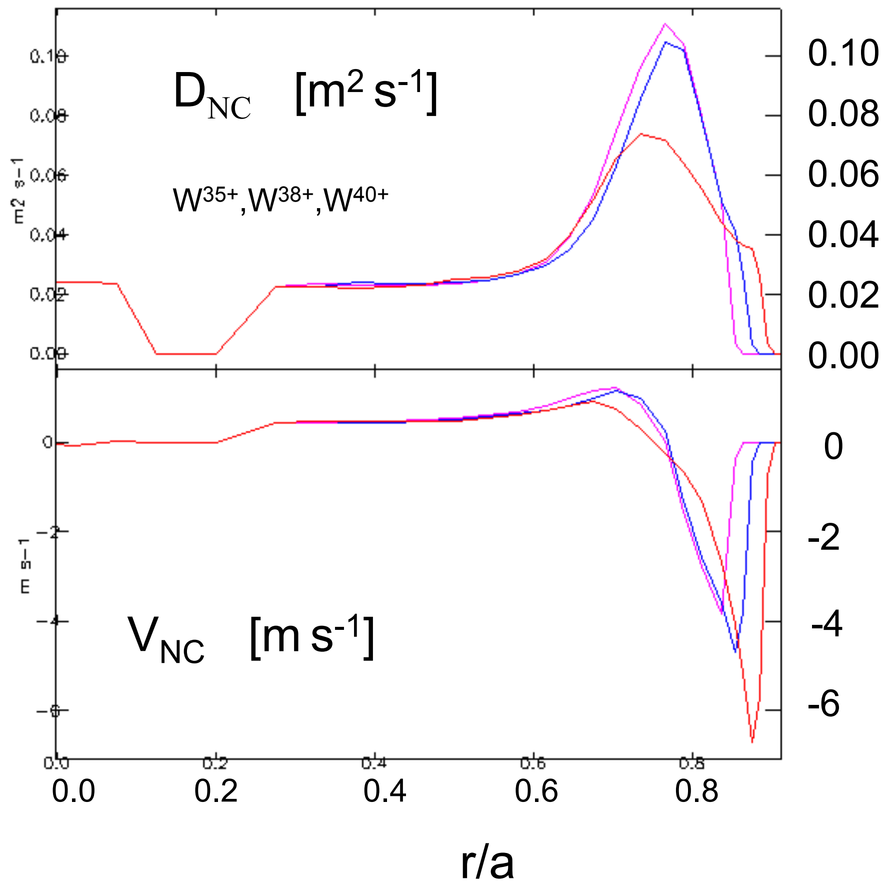




C. Angioni et al, Phys. Plasmas 22, 055902 (2015)

# Steady state plasma without rotation

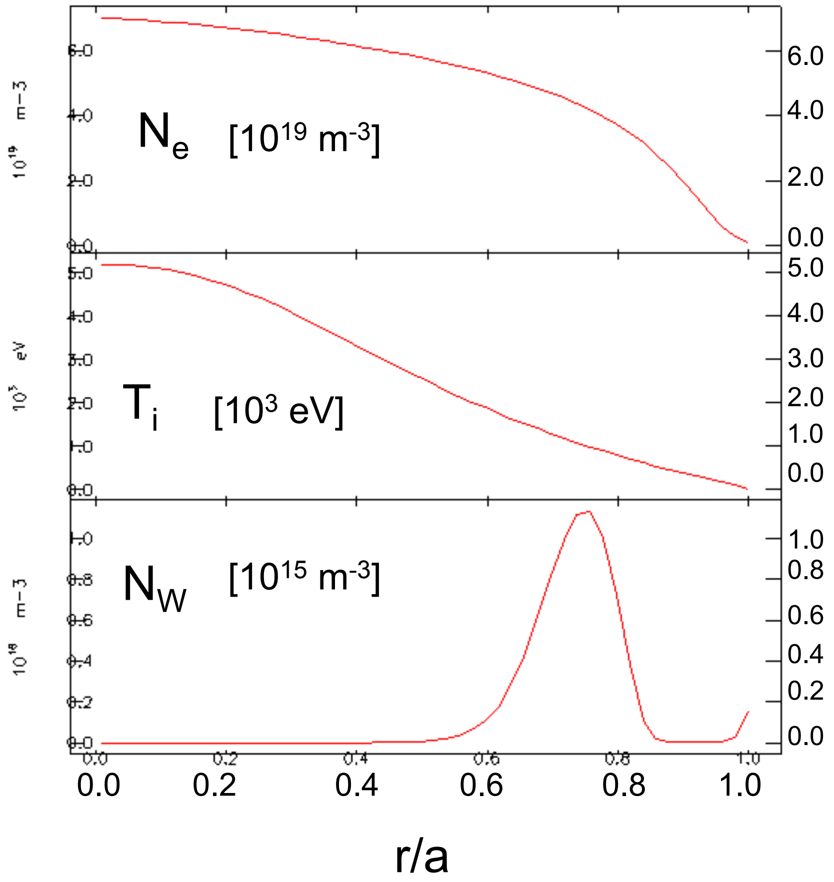
JET 82722,  W evolution from 45.0s to 49.0 s with **t=46.0s background plasma** profiles fixed

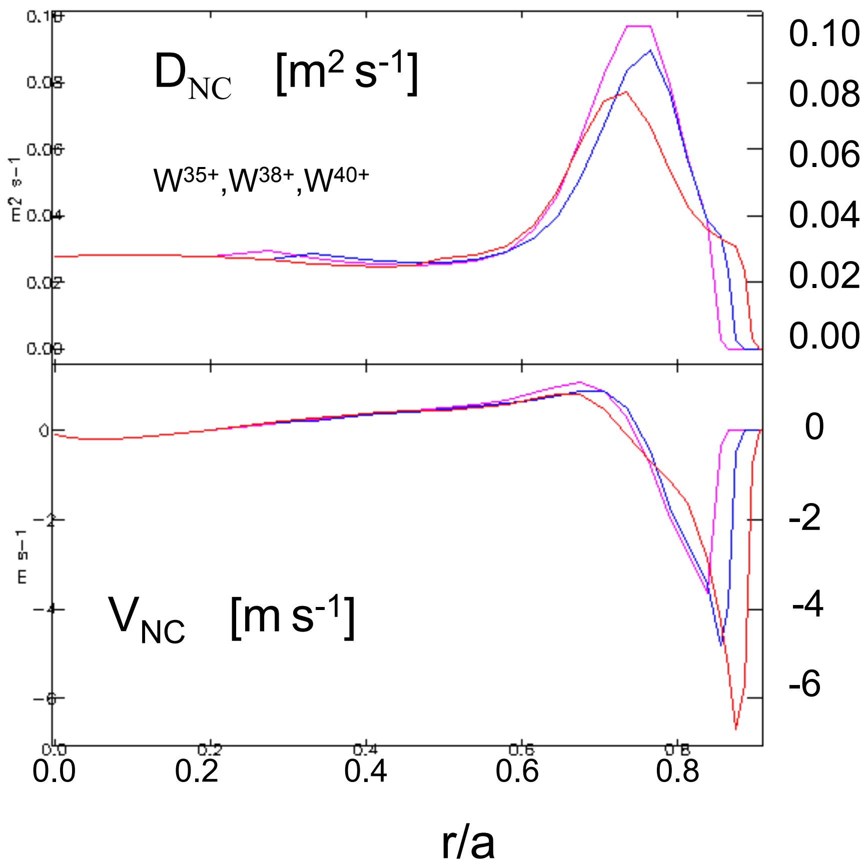


# Steady state plasma without rotation

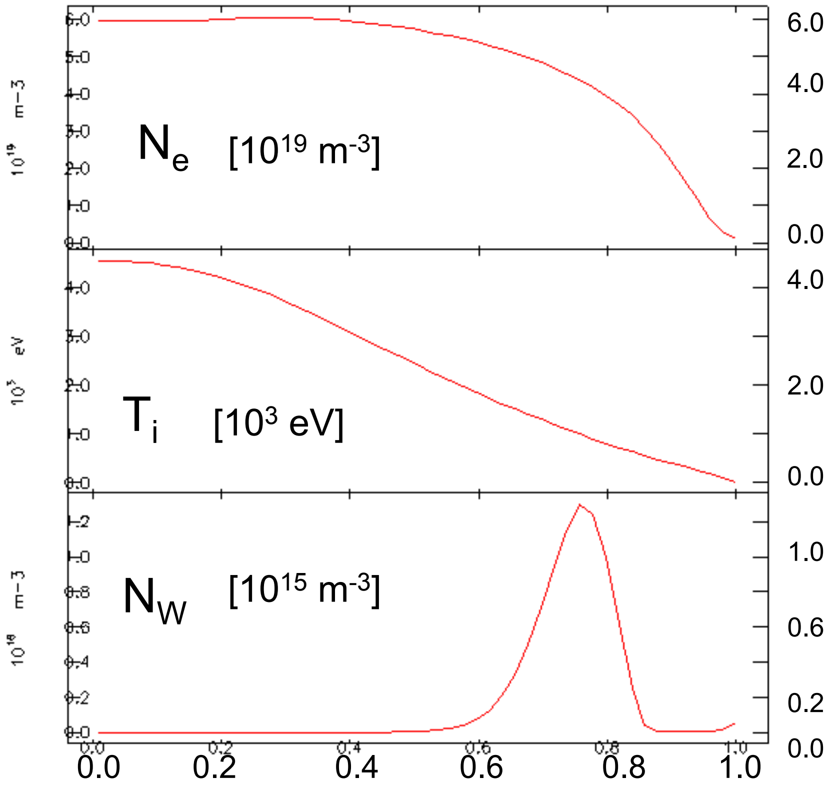
JET 82722, W evolution from 45.0s to 49.0 s with **t=47.5s background plasma** profiles fixed



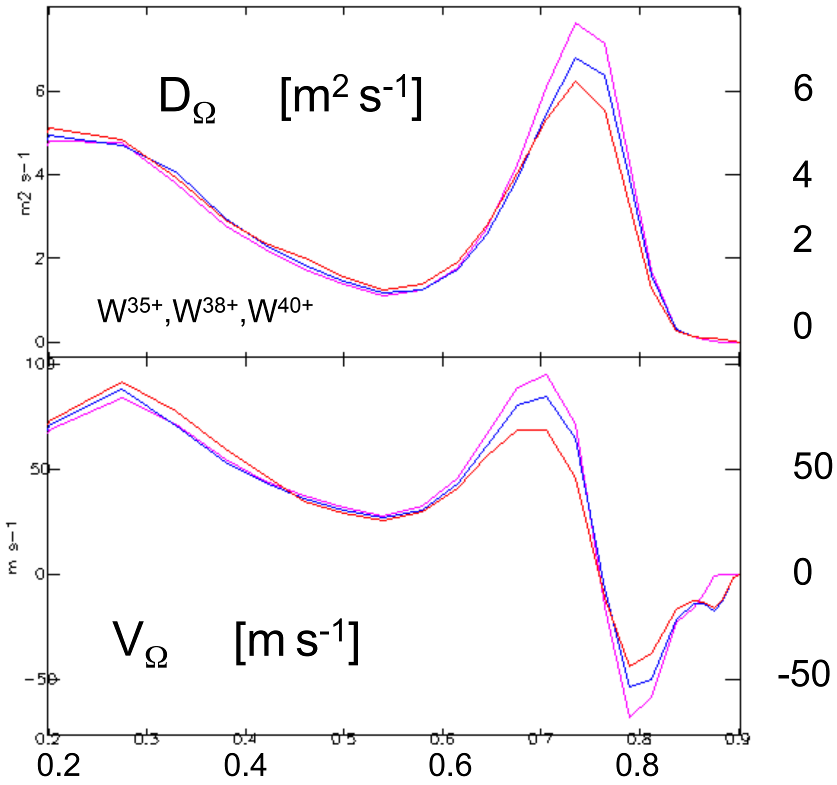



# Steady state plasma with rotation

JET 82722,  W evolution from 45.0s to 49.0 s with **t=46.0s background plasma** profiles fixed , rotation effect included



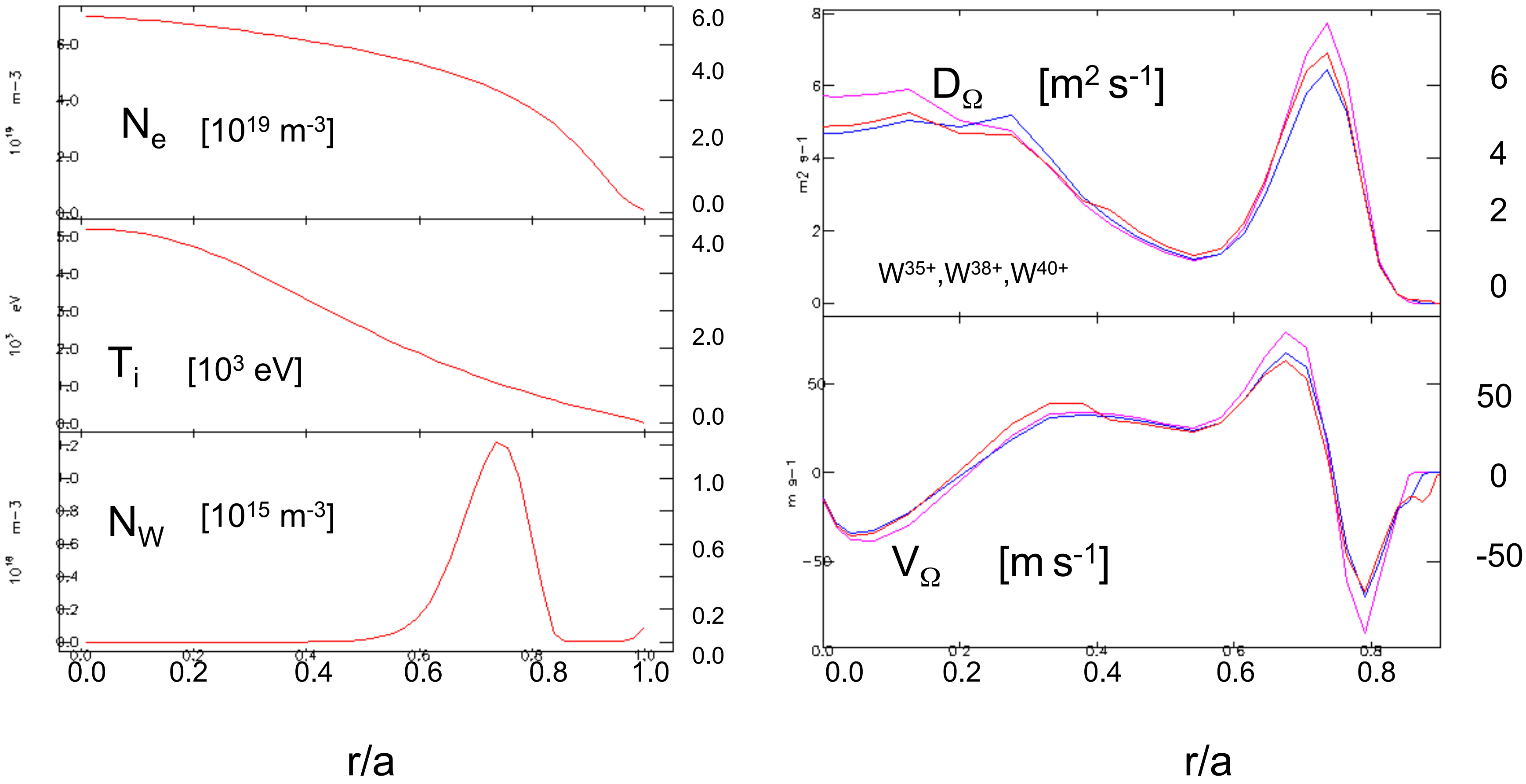

r/a




r/a

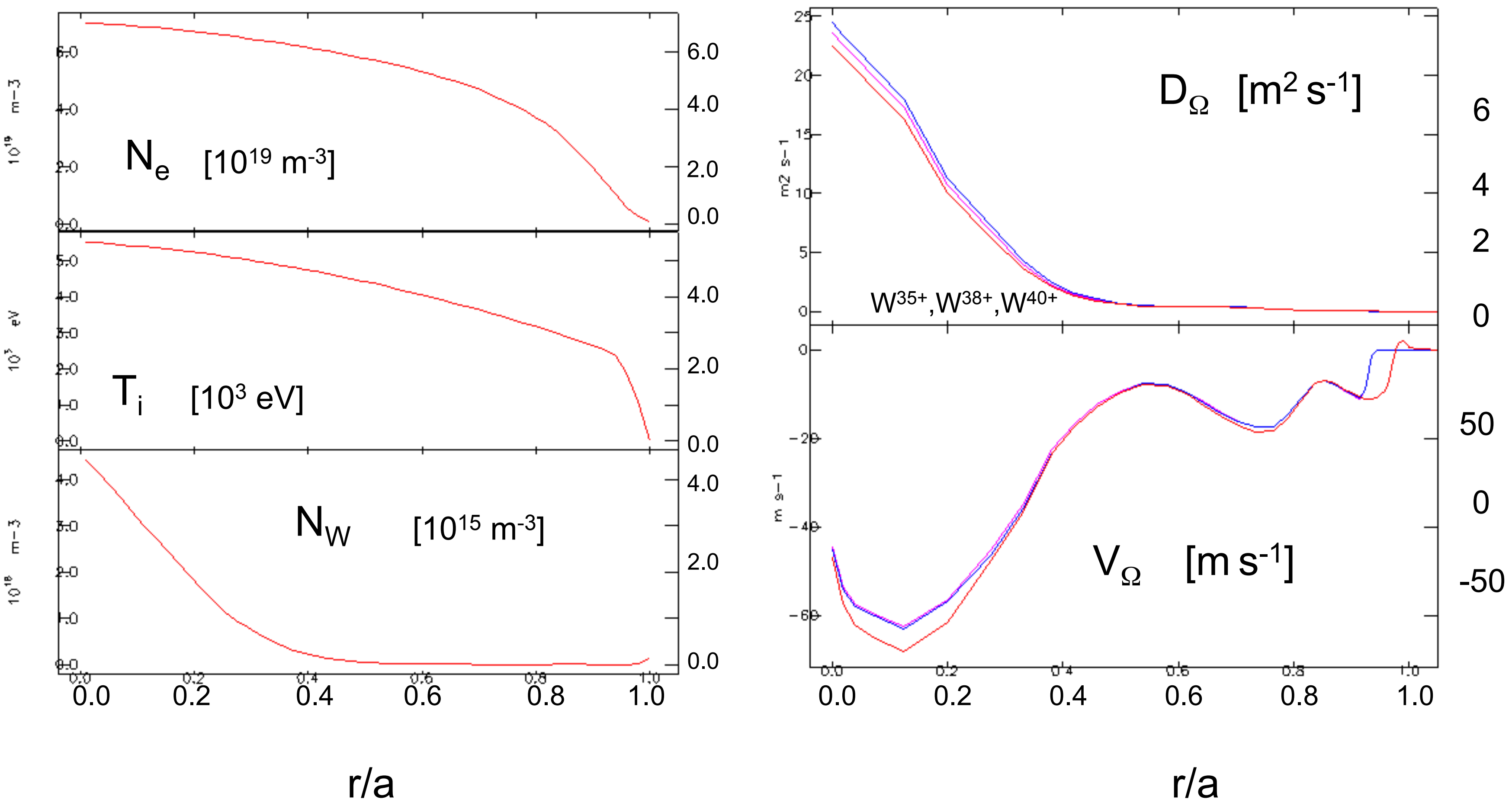
# Steady state plasma with rotation

JET 82722,  W evolution from 45.0s to 49.0 s with **t=47.5s background plasma** profiles fixed , rotation effect included



# Steady state plasma with rotation

JET 82722, W evolution from 45.0s to 49.0 s with **t=47.5s background plasma** profiles fixed , flatter $T_i$ and $T_e$, rotation effect included



# References

[1] P. Helander and D.J. Sigmar, Collisional Transport in Magnetized Plasma, Cambridge University Press

[2] J. Wesson, Tokamaks, Clarendon Press, Oxford

[3] P. H. Rutherford (1974), Physics of Fluids, 17 (9) pp. 1782

[4] M. Romanelli, M Ottaviani (1998), Plasma Physics and Controlled Fusion, **40** pp. 1767

**Recent papers on W transport**

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[6] C. Angioni *et al*, Nucl. Fusion **54**, 083028 (2014)

[7] C. Angioni and P. Helander, plasma Phys. Control. Fusion **56** (2014) 124001