Introduction to gyrokinetic variational principles

Part I: How to derive a gyrokinetic variational principle

J. W. Burby
Courant Institute of Mathematical Sciences

What is gyrokinetics?

- A kinetic model of strongly-magnetized plasma that governs dynamics on time scales long compared with particle gyroperiods

What is a gyrokinetic variational principle?

$$\delta \int L \, dt = 0$$

Variational principle

$$\Downarrow$$

$$\nabla \cdot (\epsilon_o \boldsymbol{E} + \boldsymbol{P}) = \rho_{\mathrm{gy}}$$

$$\partial_t F + \dot{\boldsymbol{X}} \cdot \nabla F + \dot{v}_{\|} \partial_{v_{\|}} F = 0$$

Gyrokinetic Vlasov-Poisson system

Why does anyone care about gyrokinetic variational principles?

- Variational approximations preserve exact conservation laws
  - Useful property for constructing:
    - Full-f collisionless gyrokinetic models
    - δf collisionless gyrokinetic models
    - Collisional gyrokinetic models
- Variational principles can be used to develop structure-preserving simulation algorithms
  - Please speak with Professor Qin about GAPS

Are gyrokinetic variational principles too complicated for you to understand?

NO

In this lecture, I will:

I) Explain how to derive gyrokinetic variational principles using a **toy model**
II) Show how to derive the drift kinetic Vlasov-Poisson system from a well-known drift kinetic Lagrangian

Gyrokinetic variational principles are derived in the following manner

Step #1: identify a (collisionless) particle-space kinetic system

This is "two-oscillator kinetics" (TOK)

$$\dot{\mu} = \frac{2g\varphi\mu}{\omega_c}\sin 2\theta$$

$$\dot{\theta} = \omega_c + \frac{2g\varphi}{\omega_c}\cos^2\theta$$

$$\ddot{\varphi} = -\Omega^2\varphi - \frac{2g\mu}{\omega_c}\cos^2\theta$$

TOK is analagous to Vlasov-Maxwell

$$\mu \sim \text{magnetic moment}$$

$$\theta \sim \text{gyrophase}$$

$$\varphi \sim \text{electric field}$$

Step #2: find a scaled variational principle for particle-space theory

“scaling” means introducing dimensionless variables

In full-on kinetic theory

$$
\begin{aligned}
\boldsymbol{v} &\rightarrow v_{\mathrm{th}}\bar{\boldsymbol{v}} \\
\boldsymbol{x} &\rightarrow L\bar{\boldsymbol{x}} \\
t &\rightarrow T\bar{t} \\
\boldsymbol{B}(\boldsymbol{x},t) &\rightarrow B_o\bar{\boldsymbol{B}}(\bar{\boldsymbol{x}},\bar{t}) \\
\boldsymbol{E}(\boldsymbol{x},t) &\rightarrow B_o v_{\mathrm{th}}\bar{\boldsymbol{E}}(\bar{\boldsymbol{x}},\bar{t})
\end{aligned}
$$

“scaling” means introducing dimensionless variables

In two-oscillator kinetics

$$\begin{aligned} \omega_c &\rightarrow \omega_c/\epsilon \\ g &\rightarrow g/\epsilon \end{aligned}$$

A correct scaling reflects the extreme speed of gyromotion

$$
\begin{aligned}
\dot{\mu} &= \frac{2g\varphi\mu}{\omega_c}\sin 2\theta \\
\dot{\theta} &= \frac{\omega_c}{\epsilon} + \frac{2g\varphi}{\omega_c}\cos^2\theta \\
\ddot{\varphi} &= -\Omega^2\varphi - \frac{2g\mu}{\omega_c}\cos^2\theta
\end{aligned}
$$

A correct scaling reflects the extreme speed of gyromotion

$$\epsilon \longrightarrow 0$$

A correct scaling reflects the extreme speed of gyromotion

$$\dot{\mu} = 0$$
$$\dot{\theta} = \frac{\omega_c}{\epsilon}$$
$$\ddot{\varphi} = 0$$

As ε tends to zero,
All dynamics freeze
Except for gyromotion.

Scaled TOK can be derived from the following Lagrangian

$$L = \mu\dot{\theta} - \left(\mu\frac{\omega_c}{\epsilon} + \frac{2g\varphi\mu}{\omega_c}\cos^2\theta\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2$$

Scaled TOK can be derived from the following Lagrangian

$$L = \mu\dot{\theta} - \left(\mu\frac{\omega_c}{\epsilon} + \frac{2g\varphi\mu}{\omega_c}\cos^2\theta\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2$$

Particle phase-space Lagrangian

Scaled TOK can be derived from the following Lagrangian

$$L = \mu\dot{\theta} - \left(\mu\frac{\omega_c}{\epsilon} + \frac{2g\varphi\mu}{\omega_c}\cos^2\theta\right) + \underbrace{\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2}_{\text{Free field Lagrangian}}$$

Varying the action leads to Euler-Lagrange equations

$$\delta \int L\, dt = 0 \quad \Rightarrow \quad \begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\mu}} - \frac{\partial L}{\partial \mu} &= 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} &= 0 \end{aligned}$$

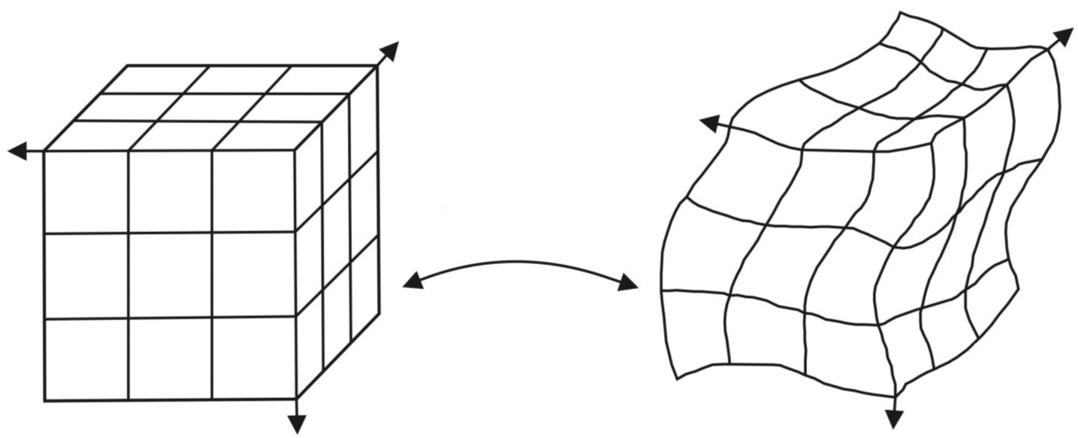
Euler-Lagrange equations lead to TOK

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\mu}} - \frac{\partial L}{\partial \mu} &= 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} &= 0
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\dot{\mu} &= \frac{2g\varphi\mu}{\omega_c}\sin 2\theta \\
\dot{\theta} &= \frac{\omega_c}{\epsilon} + \frac{2g\varphi}{\omega_c}\cos^2\theta \\
\ddot{\varphi} &= -\Omega^2\varphi - \frac{2g\mu}{\omega_c}\cos^2\theta
\end{aligned}
$$

Step #3: eliminate gyrophase from Lagrangian via gauge transformation

Gauge transformation: (coordinate change)
+(addition of total time derivatives to L)

We can change coordinates and add total derivatives to L without changing physics



We can change coordinates and add total derivatives to L without changing physics

$$\delta \int L\,dt = 0 \Leftrightarrow \delta \int (L + \dot{S})\,dt = 0$$

Let's do both and try to eliminate gyrophase from the Lagrangian!

$$\begin{aligned} \mu &= \bar{\mu} + \epsilon\,\delta\mu(\bar{\mu}, \bar{\theta}, \varphi, \dot{\varphi}, \ddot{\varphi}, \ldots) \\ \theta &= \bar{\theta} + \epsilon\,\delta\theta(\bar{\mu}, \bar{\theta}, \varphi, \dot{\varphi}, \ddot{\varphi}, \ldots) \end{aligned}$$

Near-identity transformation of particle phase-space variables

- Why near-identity ? Largest terms in Lagrangian do not depend on θ

Let's do both and try to eliminate gyrophase from the Lagrangian!

$$\bar{L} \quad = \quad L + \epsilon \frac{d}{dt} S(\bar{\mu}, \bar{\theta}, \varphi, \dot{\varphi}, \ddot{\varphi}, \ldots)$$

Total time derivatives will be added at will

- Why bother with S? Will be clear soon.

We want to find

$$\delta\mu, \delta\theta, S$$

Such that

$$\bar{L}$$

Does not depend on

$$\bar{\theta}$$

The original Lagrangian is given by

$$L = \mu\dot{\theta} - \left(\mu\frac{\omega_c}{\epsilon} + \frac{2g\varphi\mu}{\omega_c}\cos^2\theta\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2$$

After gauge transformation, the Lagrangian becomes:

$$\begin{aligned}
\bar{L} &= \bar{\mu}\,\dot{\bar{\theta}} - \left(\bar{\mu}\frac{\omega_c}{\epsilon} + \frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta}\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2 \\
&- \left(\delta\mu\,\omega_c + \epsilon\frac{2g\varphi\,\delta\mu}{\omega_c}\cos^2\bar{\theta} - \epsilon\frac{2g\varphi\bar{\mu}}{\omega_c}\sin 2\bar{\theta}\,\delta\theta\right) \\
&+ \epsilon\left(\bar{\mu}\frac{d}{dt}\delta\theta + \delta\mu\,\dot{\bar{\theta}}\right) + \epsilon\frac{dS}{dt} + O(\epsilon^2)
\end{aligned}$$

By redefining S, time derivative of transformation can be killed

$$S \to S - \frac{d}{dt}(\bar{\mu}\,\delta\theta)$$

By redefining S, time derivative of transformation can be killed

$$\begin{aligned}\bar{L} &= \bar{\mu}\,\dot{\bar{\theta}} - \left(\bar{\mu}\frac{\omega_c}{\epsilon} + \frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta}\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2 \\ &- \left(\delta\mu\,\omega_c + \epsilon\frac{2g\varphi\,\delta\mu}{\omega_c}\cos^2\bar{\theta} - \epsilon\frac{2g\varphi\bar{\mu}}{\omega_c}\sin 2\bar{\theta}\,\delta\theta\right) \\ &+ \boxed{\epsilon\left(\delta\mu\,\dot{\bar{\theta}} - \delta\theta\,\dot{\bar{\mu}}\right)} + \epsilon\frac{dS}{dt} + O(\epsilon^2)\end{aligned}$$

By relating S to δμ,δϑ, time derivatives of new coordinates can be killed

$$\delta\mu = -\frac{\partial S}{\partial \bar{\theta}}$$

$$\delta\theta = \frac{\partial S}{\partial \bar{\mu}}$$

By relating S to δμ,δϑ, time derivatives of new coordinates can be killed

$$
\begin{aligned}
\bar{L} &= \bar{\mu}\,\dot{\bar{\theta}} - \left(\bar{\mu}\frac{\omega_c}{\epsilon} + \frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta}\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2 \\
&- \left(\delta\mu\,\omega_c + \epsilon\frac{2g\varphi\,\delta\mu}{\omega_c}\cos^2\bar{\theta} - \epsilon\frac{2g\varphi\bar{\mu}}{\omega_c}\sin 2\bar{\theta}\,\delta\theta\right) \\
&+ \epsilon\left(\delta\mu\,\dot{\bar{\theta}} - \delta\theta\,\dot{\bar{\mu}}\right) + \epsilon\frac{dS}{dt} + O(\epsilon^2)
\end{aligned}
$$

By relating S to δμ,δϑ, time derivatives of new coordinates can be killed

$$
\begin{aligned}
\bar{L} &= \bar{\mu}\,\dot{\bar{\theta}} - \left(\bar{\mu}\frac{\omega_c}{\epsilon} + \frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta}\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2 \\
&- \left(-\,\omega_c\,\partial_{\bar{\theta}}S - \epsilon\frac{2g\varphi\,\partial_{\bar{\theta}}S}{\omega_c}\cos^2\bar{\theta} - \epsilon\frac{2g\varphi\bar{\mu}}{\omega_c}\sin 2\bar{\theta}\,\partial_{\bar{\mu}}S\right) \\
&+ \epsilon\partial_t S + O(\epsilon^2)
\end{aligned}
$$

$$
\partial_t S = \partial_\varphi S\,\dot{\varphi} + \partial_{\dot{\varphi}}S\,\ddot{\varphi} + \partial_{\ddot{\varphi}}S\,\dddot{\varphi} + \ldots
$$

S can now be chosen to eliminate the largest gyrophase dependent terms

$$\begin{aligned}
\bar{L}_{-1} &= -\bar{\mu}\,\omega_c \\
\bar{L}_0 &= \bar{\mu}\,\dot{\bar{\theta}} - \boxed{\left(\frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta} - \omega_c\,\partial_{\bar{\theta}}S\right)} + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2 \\
\bar{L}_1 &= \frac{2g\varphi\partial_{\bar{\theta}}S}{\omega_c}\cos^2\bar{\theta} + \frac{2g\varphi\,\bar{\mu}}{\omega_c}\sin 2\bar{\theta}\,\partial_{\bar{\mu}}S + \partial_t S \\
\bar{L}_2 &= \ldots
\end{aligned}$$

S can now be chosen to eliminate the largest gyrophase dependent terms

$$\frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta} - \omega_c\,\partial_{\bar{\theta}}S = \langle\frac{2g\varphi\bar{\mu}}{\omega_c}\cos^2\bar{\theta}\rangle$$

$$\Rightarrow \qquad \omega_c\,\partial_{\bar{\theta}}S = \frac{g\varphi\bar{\mu}}{\omega_c}\cos 2\bar{\theta}$$

$$\Rightarrow \qquad S = \frac{g\varphi\bar{\mu}}{2\omega_c^2}\sin 2\theta$$

With S chosen, new coordinates completely specified…

$$\delta\mu = -\frac{g\varphi\bar{\mu}}{\omega_c^2}\cos 2\bar{\theta}$$

$$\delta\theta = \frac{g\varphi}{2\omega_c^2}\sin 2\bar{\theta}$$

…and Lo is now gyrophase independent…

$$\bar{L}_0 = \bar{\mu}\,\dot{\bar{\theta}} - \frac{g\varphi\bar{\mu}}{\omega_c} + \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\Omega^2\varphi^2$$

…but sadly L1,L2,… still depend on gyrophase!

$$\bar{L}_1 = \frac{g^2\varphi^2\bar{\mu}}{\omega_c^3} + \frac{g^2\varphi^2\bar{\mu}}{\omega_c^3}\cos 2\bar{\theta} + \frac{g\dot{\varphi}\bar{\mu}}{2\omega_c^2}\sin 2\bar{\theta}$$

Q: How can we eliminate the remaining gyrophase dependence?

A: apply a second coordinate transformation!

By repeating this procedure, gyrophase dependence can be pushed to arbitrarily-high order

$$\bar{L}(\vartheta, M, \varphi, \dot{\vartheta}, \dot{M}, \dot{\varphi}, \ddot{\varphi}, ...) =$$

$$M\dot{\vartheta} - (\epsilon^{-1}\, h_{-1} + h_0 + \epsilon\, h_1 + ...) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2$$

$$
\begin{aligned}
h_{-1} &= M\omega_c \\
h_0 &= M\frac{g\varphi}{\omega_c} \\
h_1 &= -M\frac{g^2\varphi^2}{2\omega_c^3} \\
h_2 &= M\frac{g^3\varphi^3}{2\omega_c^5} \\
h_3 &= -M\left(\frac{5g^4\varphi^4}{8\omega_c^7} + \frac{g^2\dot{\varphi}^2}{8\omega_c^5}\right) \\
h_4 &= -M\left(\frac{7g^5\varphi^5}{8\omega_c^9} + \frac{3g^3\varphi\dot{\varphi}^2}{8\omega_c^7} - \frac{g^3\varphi^2\ddot{\varphi}}{8\omega_c^7}\right) \\
h_5 &= M\left(\frac{21g^6\varphi^6}{16\omega_c^{11}} + \frac{43g^4\varphi^2\dot{\varphi}^2}{32\omega_c^9} - \frac{9g^4\varphi^3\ddot{\varphi}}{32\omega_c^9} + \frac{g^2\ddot{\varphi}^2}{32\omega_c^7}\right)
\end{aligned}
$$

Step #4 (final step): truncate!

We don't know all of the h's, so we must choose finitely many to keep

- Suppose we keep just h_{-1} , h_0

$$\bar{L} = M\dot{\vartheta} - \left(\epsilon^{-1} M\omega_c + M\frac{g\varphi}{\omega_c}\right) + \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\Omega^2\varphi^2$$

$$\begin{aligned} \dot{M} &= 0 \\ \dot{\vartheta} &= \epsilon^{-1}\omega_c + \frac{g\varphi}{\omega_c} \\ \ddot{\varphi} &= -\Omega^2\varphi - \frac{Mg}{\omega_c} \end{aligned}$$

Magnetic moment and energy are conserved

$$
\begin{aligned}
E &= \dot{\vartheta}\partial_{\dot{\vartheta}}\bar{L} + \dot{\varphi}\partial_{\dot{\varphi}}\bar{L} - \bar{L} \\
&= \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\Omega^2\varphi^2 + \epsilon^{-1}M\omega_c + M\frac{g\varphi}{\omega_c}
\end{aligned}
$$

Exercise: what is the conserved energy if we keep $h_{-1}, h_0, h_1, h_2, h_3, h_4, h_5$?

Summary of how to derive gyrokinetic variational principles

- Step #1: identify a (collisionless) particle-space kinetic system

- Step #2: find a scaled variational principle for particle-space theory

- Step #3: eliminate gyrophase from Lagrangian via gauge transformation

- Step #4 (final step): truncate!

Introduction to gyrokinetic variational principles

Part II: How to wield a gyrokinetic variational principle

J. W. Burby
Courant Institute of Mathematical Sciences

Here is a "real" full-F drift kinetic Lagrangian

$$
\begin{aligned}
L(F, V, \varphi) \quad &= \quad \sum_s \int \ell_s(z, V_s(z), \varphi)\, F_s(z)\, d\Omega_s(z) \\
&\quad + \frac{1}{2}\epsilon_o \int |\nabla \varphi|^2\, d^3\boldsymbol{X}
\end{aligned}
$$

The dynamical fields are

$$V \sim \text{Eulerian phase-space fluid velocity}$$
$$F \sim \text{Gyrocenter distribution function}$$
$$\varphi \sim \text{electrostatic field}$$

The gyrocenter Lagrangian is given by

$$\ell_s(z, \dot{z}, \varphi) = q_s \boldsymbol{A}_s^* \cdot d\boldsymbol{X} + \frac{m_s}{q_s}\mu\dot{\zeta} - q_s\varphi - K_s(\boldsymbol{E})$$

$$z = (\boldsymbol{X}, v_\parallel, \mu, \zeta)$$

$$\boldsymbol{A}_s^* = \boldsymbol{A} + \frac{m_s}{q_s} v_\parallel \hat{b} - \frac{m_s}{q_s^2}\mu \boldsymbol{W}$$

$$K_s(\boldsymbol{E}) = \mu B + \frac{1}{2} m_s v_\parallel^2 - \frac{1}{2} m_s\, u_E^2$$

$$\boldsymbol{u}_E = \frac{\boldsymbol{E} \times \hat{b}}{B} \qquad \boldsymbol{W} = (\nabla \hat{e}_1) \cdot \hat{e}_2 + \frac{\hat{b} \cdot \nabla \times \hat{b}}{2} \hat{b}$$

I will show you how to derive the drift kinetic Vlasov-Poisson system from this Lagrangian

It will be useful to introduce two mathematical concepts

1. Phase space and configuration space inner products
2. Functional derivatives

There is an inner product on the space of phase space functions

$$\langle g, h \rangle_s = \int g(z)\, h(z)\, d\Omega_s(z)$$

$$d\Omega(z) = \mathcal{J}_s(z)\, d^3\boldsymbol{X}\, dv_{||}\, d\mu\, d\zeta$$

$$\mathcal{J}_s = \frac{B^*_{||s}}{m_s} = \frac{\hat{b} \cdot \nabla \times \boldsymbol{A}^*_s}{m_s}$$

And also on the spaces of vector fields and functions on configuration space

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int \boldsymbol{u} \cdot \boldsymbol{v} \, d^3\boldsymbol{X}$$

$$\langle \varphi, \psi \rangle = \int \varphi \, \psi \, d^3\boldsymbol{X}$$

In terms of these inner products, the Lagrangian is

$$L = \sum_s \langle \ell_s, F_s \rangle_s + \frac{1}{2}\epsilon_o \langle \boldsymbol{E}, \boldsymbol{E} \rangle$$

Functional derivatives are closely related to first variations

$$\left.\frac{d}{d\epsilon}\right|_0 L(F + \epsilon\,\delta F, V + \epsilon\,\delta V, \varphi + \epsilon\,\delta\varphi)$$

$$\equiv \sum_s \left(\left\langle \frac{\delta L}{\delta V_s^i}, \delta V_s^i \right\rangle_s + \left\langle \frac{\delta L}{\delta F_s}, \delta F_s \right\rangle_s \right) + \left\langle \frac{\delta L}{\delta \varphi}, \delta\varphi \right\rangle$$

The first fixed-endpoint variation of the action is given by

$$\delta S = \int_{t_1}^{t_2} \left[\sum_s \left(\left\langle \frac{\delta L}{\delta V_s^i}, \delta V_s^i \right\rangle_s + \left\langle \frac{\delta L}{\delta F_s}, \delta F_s \right\rangle \right) + \left\langle \frac{\delta L}{\delta \varphi}, \delta \varphi \right\rangle \right] dt$$

Key point: the variations of V,F are constrained!

$$\delta V_s^i = \dot{\xi}_s^i + V_s^j \partial_j \xi_s^i - \xi_s^j \partial_j V_s^i$$

$$\delta F_s = -\mathcal{J}_s^{-1} \partial_i (\mathcal{J}_s \, \xi_s^i \, F_s)$$

$$\xi \sim \text{arbitrary vector field on phase space}$$

Why? See literature on "Euler-Poincaré variational principles."

These constrained variations can simplify δS using the following identities

$$\langle g, \partial_i h \rangle_s = -\langle \mathcal{J}_s^{-1} \partial_i (\mathcal{J}_s g), h \rangle$$

$$\begin{aligned}
\langle g, \partial_i h \rangle_s &= \int g \, \partial_i h \, \mathcal{J}_s \, d^3\boldsymbol{X} \, dv_{||} \, d\mu \, d\zeta \\
&= -\int \partial_i (\mathcal{J}_s \, g) \, h \, d^3\boldsymbol{X} \, dv_{||} \, d\mu \, d\zeta \\
&= -\int \mathcal{J}_s^{-1} \partial_i (\mathcal{J}_s \, g) \, h \, \mathcal{J}_s \, d^3\boldsymbol{X} \, dv_{||} \, d\mu \, d\zeta \\
&= -\langle \mathcal{J}_s^{-1} \partial_i (\mathcal{J}_s \, g), h \rangle_s
\end{aligned}$$

The variation of S simplifies to

$$\delta S$$

$$= \int_{t_1}^{t_2} \left[\sum_s \left\langle E_i L_s , \xi_s^i \right\rangle_s + \left\langle \frac{\delta L}{\delta \varphi}, \delta \varphi \right\rangle \right]$$

The variation of S simplifies to

$$
\begin{aligned}
E_i L_s \quad = \quad & -\frac{d}{dt}\frac{\delta L}{\delta V_s^i} - \frac{\delta L}{\delta V_s^j}\partial_i V_s^j \\
& -\mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) \\
& +F_s \partial_i \frac{\delta L}{\delta F_s}
\end{aligned}
$$

The Euler-Lagrange equations are therefore

$$\frac{d}{dt}\frac{\delta L}{\delta V_s^i} + \partial_i V_s^j \frac{\delta L}{\delta V_s^j} + \mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) = F_s \partial_i \frac{\delta L}{\delta F_s}$$

$$\frac{\delta L}{\delta \varphi} = 0$$

Let's see how to recover the usual Euler-Lagrange equations for gyrocenters

$$\frac{d}{dt}\frac{\delta L}{\delta V_s^i} + \partial_i V_s^j \frac{\delta L}{\delta V_s^j} + \mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) = F_s \partial_i \frac{\delta L}{\delta F_s}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \ell_s}{\partial \dot{z}^i} - \frac{\partial \ell_s}{\partial z^i} = 0 \;?$$

Consider varying L w.r.t V

$$
\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_0 L(F, V + \epsilon\,\delta V, \varphi) &= \sum_s \int \left.\frac{d}{d\epsilon}\right|_0 \ell_s(z, [V_s + \epsilon\,\delta V_s](z), \varphi)\, F_s(z)\, d\Omega_s(z) \\
&= \sum_s \int \frac{\partial \ell_s}{\partial \dot{z}^i}(z, V_s(z), \varphi)\, \delta V_s^i(z)\, F_s(z)\, d\Omega_s(z) \\
&= \sum_s \left\langle F_s \frac{\delta \tilde{\ell}_s}{\delta \dot{z}}, \delta V_s^i \right\rangle
\end{aligned}
$$

The functional derivative of L w.r.t. V is therefore

$$\Rightarrow \frac{\delta L}{\delta V_s^i}(z) = F_s(z)\frac{\delta \tilde{\ell}_s}{\delta \dot{z}^i}(z) \equiv F_s(z)\frac{\partial \ell_s}{\partial \dot{z}^i}(z, V_s(z), \varphi)$$

Consider varying L w.r.t F

$$
\begin{aligned}
\frac{d}{d\epsilon}\bigg|_0 L(F+\epsilon\,\delta F, V, \varphi) &= \frac{d}{d\epsilon}\bigg|_0 \sum_s \int \ell_s(z, V_s(z), \varphi)(F_s+\epsilon\,\delta F_s)(z)\, d\Omega_s(z) \\
&= \sum_s \int \tilde{\ell}_s(z)\delta F_s(z)\, d\Omega_s(z) \\
&= \sum_s \langle \tilde{\ell}_s, \delta F_s \rangle_s
\end{aligned}
$$

The functional derivative of L w.r.t. F is therefore

$$\Rightarrow \frac{\delta L}{\delta F_s}(z) = \tilde{\ell}_s(z) \equiv \ell_s(z, V_s(z), \varphi)$$

Now we can calculate each term in the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\delta L}{\delta V_s^i} + \partial_i V_s^j \frac{\delta L}{\delta V_s^j} + \mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) = F_s \partial_i \frac{\delta L}{\delta F_s}$$

$$\begin{aligned} \frac{d}{dt}\frac{\delta L}{\delta V_s^i} &= \frac{d}{dt}\left(F_s \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i}\right) \\ &= \partial_t F_s \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i} + F_s \frac{d}{dt}\frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i} \end{aligned}$$

$$\frac{d}{dt}\frac{\delta L}{\delta V_s^i} + \underbrace{\partial_i V_s^j \frac{\delta L}{\delta V_s^j}} + \mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) = F_s \partial_i \frac{\delta L}{\delta F_s}$$

$$\frac{\delta L}{\delta V_s^j}\partial_i V_s^j \quad = \quad F_s \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^j}\partial_i V_s^j$$

$$\frac{d}{dt}\frac{\delta L}{\delta V_s^i} + \partial_i V_s^j \frac{\delta L}{\delta V_s^j} + \underbrace{\mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right)} = F_s \partial_i \frac{\delta L}{\delta F_s}$$

$$\begin{aligned}
\mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) &= \mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j F_s \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i}\right) \\
&= \mathcal{J}_s^{-1}\partial_j(\mathcal{J}_s V_s^j F_s)\frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i} + F_s V_s^j \partial_j \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i}
\end{aligned}$$

$$\frac{d}{dt}\frac{\delta L}{\delta V_s^i} + \partial_i V_s^j \frac{\delta L}{\delta V_s^j} + \mathcal{J}_s^{-1}\partial_j\left(\mathcal{J}_s V_s^j \frac{\delta L}{\delta V_s^i}\right) = F_s \partial_i \frac{\delta L}{\delta F_s}$$

$$\begin{aligned} F_s \partial_i \frac{\delta L}{\delta F_s} &= F_s \partial_i \tilde{\ell}_s \\ &= F_s \frac{\partial \tilde{\ell}_s}{\partial z^i} + F_s \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^j} \partial_j V_s^i \end{aligned}$$

Adding these results together gives

$$\left[\partial_t F_s + \mathcal{J}_s^{-1}\partial_j(\mathcal{J}_s V_s^j F_s)\right]\frac{\partial\tilde{\ell}_s}{\partial\dot{z}^i} + F_s\left(\frac{d}{dt} + V_s^j\partial_j\right)\frac{\partial\tilde{\ell}_s}{\partial\dot{z}^i} = F_s\frac{\partial\tilde{\ell}_s}{\partial z^i}$$

Adding these results together gives

$$\left[\partial_t F_s + \mathcal{J}_s^{-1}\partial_j(\mathcal{J}_s V_s^j F_s)\right]\frac{\partial\tilde{\ell}_s}{\partial\dot{z}^i} + F_s\left(\frac{d}{dt} + V_s^j\partial_j\right)\frac{\partial\tilde{\ell}_s}{\partial\dot{z}^i} = F_s\frac{\partial\tilde{\ell}_s}{\partial z^i}$$

Would be zero
if F satisfied the
kinetic equation

Adding these results together gives

$$\left[\partial_t F_s + \mathcal{J}_s^{-1}\partial_j(\mathcal{J}_s V_s^j F_s)\right]\frac{\partial\tilde{\ell}_s}{\partial\dot{z}^i} + F_s\left(\frac{d}{dt} + V_s^j\partial_j\right)\frac{\partial\tilde{\ell}_s}{\partial\dot{z}^i} = F_s\frac{\partial\tilde{\ell}_s}{\partial z^i}$$

Equivalent to usual gyrocenter E-L eq.

$$\Rightarrow \frac{d}{dt}\frac{\partial\ell_s}{\partial\dot{z}^i} - \frac{\partial\ell_s}{\partial z^i} = 0$$

What is our conclusion?

- *If* F satisfies the kinetic equation and V satisfies the single-gyrocenter Euler-Lagrange equations, the field-theoretic Euler-Lagrange equation is satisfied
  - To understand the the “only if” statement, it is necessary to delve into Euler-Poincare theory

I will leave it as an exercise to calculate the functional derivative of L w.r.t. φ

Result: the other Euler-Lagrange equation gives the gyrokinetic Poisson equation

$$\frac{\delta L}{\delta \varphi} = 0$$

$$\Downarrow$$

$$\nabla \cdot \left(\epsilon_o \boldsymbol{E} + \epsilon_o \frac{c^2}{V_A^2} \boldsymbol{E}_\perp \right) = \rho_{\rm gy}$$

The following system of equations therefore satisfies δS=0

$$\left(\frac{d}{dt} + V_s^j \partial_j\right) \frac{\partial \tilde{\ell}_s}{\partial \dot{z}^i} = \frac{\partial \tilde{\ell}_s}{\partial z^i}$$

$$\partial_t F_s + \mathcal{J}_s^{-1} \partial_j (\mathcal{J}_s V_s^j F_s) = 0$$

$$\nabla \cdot \left(\epsilon_o \boldsymbol{E} + \epsilon_o \frac{c^2}{V_A^2} \boldsymbol{E}_\perp\right) = \rho_{\rm gy}$$

$$\boldsymbol{E} = -\nabla \varphi$$

After you can derive these Euler-Lagrange equations yourself…

- You will be in a good position to:
  - Understand how Noether's theorem leads to GK conservation laws
  - Understand the Hamiltonian (as opposed to Lagrangian) formulations of gyrokinetics
  - Learn the variational formulation of the Vlasov-Maxwell system
    - Professor Qin's GAPS project used this variational principle to develop very powerful structure-preserving integrators for Vlasov-Maxwell

No more school!

No more school!